Certificate

in

Business Administration

Study Manual

Introduction to Quantitative Methods

The Association of Business Executives

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ABE Certificate in Business Administration

Study Manual

Introduction to Quantitative Methods

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Certificate in Business Administration

Introduction to Quantitative Methods

Syllabus

Aims

1. Apply the rules of numeracy.
2. Carry out basic numerical calculations with or without a calculator.
3. Use elementary algebra.
4. Present and use mathematical data in graphical form.
5. Present and analyse simple business data.
6. Apply statistical analysis to business data.
7. Use of the laws of probability.

Programme Content and Learning Objectives

After completing the programme, the student should be able to:

1. Demonstrate the role of numeracy by:

- applying the four rules to whole numbers, fractions and decimals;
- expressing numbers in standard form;
- multiplying and dividing negative numbers.

2. Apply calculations to:

- compare numbers using ratios, proportions and percentages;
- obtain values for simple financial transactions involving purchases, wages, taxation, discounts;
- calculate values using simple and compound interest;
- convert foreign currency;
- make a calculations involving roots and powers;
- evaluate terms involving a sequence of operations and use of brackets;
- interpret, transpose and evaluate formulae;
- approximate data using rounding, significant figures.

3. Use algebraic methods to:

- solve linear and simultaneous equations;
- solve quadratic equations using the factorisation and formulae;
- determine the equations of a straight line through two points;
- determine the gradient and intercept of a straight line.

4. Construct and use:

- charts and diagrams derived from tabular data;
- graphs applying general rules and principles of graphical construction including axes, choice of scale and zero. Identify points of importance e.g. maximum, minimum and break even.

5. Apply statistical methods to:

- distinguish and use sigma continuous variables;
- represent and interpret variables using histograms and cumulative frequency curves;
- recognise and use sigma notation for summation;
- calculate and interpret summary statistics: these would include measures of location (mean, mode, median), measures of dispersion (range, interquartile range, standard deviation, coefficient of variation) and measures of skewness;
- construct statistical graphs for time series, identifying trends, seasonal and random components;
- determine a trend using moving averages and make a simple forecast.

6. Apply the laws of probability to:

- mutually exclusive, independent and dependent events;
- determine probability using sample spaces, contingency tables or tree diagrams;
- determined probability using the normal distribution including use of tables.

Method of Assessment

By written examination. The pass mark is 40%. Time allowed 3 hours.

The question paper will contain TWO sections:

- Section A contains four questions on parts 1 to 4 of the programme content, of which three must be answered.
- Section B contains three questions on part 5 and 6 of the programme content, of which two must be answered.

All questions carry 20 marks.

Students may use electronic calculators for most questions, but are reminded of the importance of showing detailed steps in calculations.

Reading List

- Francis, A. (1998), *Business Mathematics and Statistics* (5th edition); D P Publications

Introduction to Quantitative Methods

Formulae for Business Mathematics and Statistics

The examiners for this subject have identified the following key formulae for the examination. You should ensure that you are familiar with both their meaning and use. (This is not to imply that other formulae are not also essential to understand and apply.)

Interest

The formula for calculating compound interest:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where: A = Accrued amount

P = Original principal

r = Rate of interest (for a particular time period, usually annual)

n = Number of time periods

Depreciation

- Straight-line method:

$$\text{Annual depreciation} = \frac{\text{Cost of asset}}{\text{Useful life}}$$

or

$$\text{Annual depreciation} = \frac{(\text{Cost of asset}) - (\text{Value at end of useful life})}{\text{Useful life}}$$

- Reducing balance method:

$$D = B(1 - i)^n$$

where: D = Depreciated value at the end of the nth time period

B = Original value at beginning of time period

i = Depreciation rate (as a proportion)

n = Number of time periods (normally years)

Straight line

A linear function is one for which, when the relationship is plotted on a graph, a straight line is obtained.

The expression of a linear function and, hence, the formula of a straight line takes the following form:

$$y = mx + c$$

Note that: c = the y-intercept (the point where the line crosses the y axis)

m = the gradient (or slope) of the line

Quadratic equation

A quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Statistical measures

- Mean for a frequency distribution:

$$\bar{x} = \frac{\Sigma(fx)}{\Sigma f}$$

- Computational formula for the standard deviation of a frequency distribution:

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

or $$= \sqrt{\frac{1}{n}\left(\sum fx^2 - \frac{(\sum fx)^2}{n}\right)}$$

where: $n = \Sigma f$

(Note that, for a grouped frequency distribution, x is the class mid-point.)

- Pearson's measure of skewness

$$Psk = \frac{\text{Mean} - \text{Mode}}{\text{Standard deviation}}$$

$$= \frac{3(\text{Mean} - \text{Median})}{\text{Standard deviation}}$$

Probability

- Probability rules

  Probability limits: $0 \le P(A) \le 1$

  Total probability rule: $\sum P = 1$ (for all outcomes)

  Complementary rule: $P(\bar{A}) = 1 - P(A)$

- The addition rule

  If A and B are any two mutually exclusive events, then:

  $P(A \text{ or } B) = P(A) + P(B)$

- The multiplication rule

  If A and B are any two independent events, then:

  $P(A \text{ and } B) = P(A) \times P(B)$

- Definition of conditional probability (equally likely outcome set):

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)}$$

- Expected value of x

  If some variable x has its values specified with associated probabilities p, then:

  $$\text{Expected value of } x = \sum Px$$

Standard normal distribution

The table of values of the standard normal distribution set out on the next page provides a means of determining the probability of an observation, x, lying within specified standard deviations of the mean of the distribution (μ).

AREAS IN TAIL OF THE NORMAL DISTRIBUTION

$\frac{(x-\mu)}{\sigma}$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
3.0	.00135									

Study Unit 1

Basic Numerical Concepts

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INTRODUCTION

The skill of being able to handle numbers well is very important in many jobs. You may often be called upon to handle numbers and express yourself clearly in numerical form – and this is part of effective communication, as well as being essential to understanding and solving many business problems.

Sometimes we can express numerical information more clearly in pictures – for example, in charts and diagrams. We shall consider this later in the course. Firstly, though, we need to examine the basic processes of manipulating numbers themselves. In this study unit, we shall start by reviewing a number of basic numerical concepts and operations. It is likely that you will be familiar with at least some of these, but it is essential to ensure that you fully understand the basics before moving on to some of the more advanced applications in later units.

Throughout the course, as this is a practical subject, there will be plenty of practice questions for you to work through. To get the most out of the material, we recommend strongly that you do attempt all these questions.

One initial word about calculators. It is perfectly acceptable to use calculators to perform virtually all arithmetic operations – indeed, it is accepted practice in examinations. You should, therefore, get to know how to use one quickly and accurately. However, you should also ensure that you know exactly how to perform all the same operations manually. Understanding how the principles work is essential if you are to use the calculator correctly.

Objectives

When you have completed this study unit you will be able to:

- identify some different number systems;
- round-up numbers and correct them to significant figures;
- carry out calculations involving the processes of addition, subtraction, multiplication and division;
- deal with negative numbers;
- cancel fractions down to their simplest terms;
- change improper fractions to mixed numbers or integers, and vice versa;
- add, subtract, multiply and divide fractions and mixed numbers;
- carry out calculations using decimals;
- calculate ratios and percentages, carry out calculations using percentages and divide a given quantity according to a ratio;
- explain the terms index, power, root, reciprocal and factorial.

A. NUMBER SYSTEMS

Denary number system

We are going to begin by looking at the number system we use every day.

Because ten figures (i.e. symbols 0 to 9) are used, we call this number system the "***base ten***" system or "***denary***" system. This is illustrated in the following table.

Figure 1.1: The denary system

Seventh column	Sixth column	Fifth column	Fourth column	Third column	Second column	First column
Millions	Hundreds of thousands	Tens of thousands	Thousands	Hundreds	Tens	Units

The number of columns indicates the actual amounts involved. Consider the number 289. This means that, in this number, there are:

two hundreds (<u>2</u>89)

eight tens (2<u>8</u>9)

nine units (28<u>9</u>)

Before we look at other number systems, we can note that it is possible to have negative numbers – numbers which are less than zero. Numbers which are more than zero are positive. We indicate a negative number by a minus sign (−). A common example of the use of negative numbers is in the measurement of temperature – for example, −20°C (i.e. 20°C below zero).

We could indicate a positive number by a plus sign (+), but in practice this is not necessary and we adopt the convention that, say, 73 means +73.

Roman number system

We have seen that position is very important in the number system that we normally use. This is equally true of the Roman numerical system, but this system uses letters instead of figures. Although we invariably use the normal denary system for calculations, Roman numerals are still in occasional use in, for example, the dates on film productions, tabulation, house numbers, etc.

Here are some Roman numbers and their denary equivalent:

XIV	14	XXXV1	36
1X	9	LV111	58

Binary system

As we have said, our numbering system is a "base ten" system. By contrast, computers use "***base two***", or the "***binary***" system. This uses just two symbols (figures), 0 and 1.

In this system, counting from 0 to 1 uses up all the symbols, so a new column has to be started when counting to 2, and similarly the third column at 4.

Figure 1.2: The binary system

Denary	Binary
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001

This system is necessary because computers can only easily recognise two figures – zero or one, corresponding to components of the computer being either switched off or on. When a number is transferred to the computer, it is translated into the binary system.

B. NUMBERS – APPROXIMATION AND INTEGERS

Approximation

Every figure used in a particular number has a meaning. However, on some occasions, precision is not needed and may even hinder communication. For example, if you are calculating the cost of the journey, you need only express the distance to the nearest mile.

It is important to understand the methods of approximation used in number language.

(a) Rounding off

Suppose you are given the figure 27,836 and asked to quote it to the nearest thousand. Clearly, the figure lies between 27,000 and 28,000. The halfway point between the two is 27,500. 27,836 is more than this and so, to the nearest thousand, the answer would be 28,000.

There are strict mathematical rules for deciding whether to round up or down to the nearest figure:

Look at the digit after the one in the last required column. Then:

- ***for figures of four or less, round down;***
- ***for figures of five or more, round up.***

Thus, when rounding off 236 to the nearest hundred, the digit after the one in the last required column is 3 (the last required column being the hundreds column) and the answer would be to round down to 200. When rounding off 3,455 to the nearest ten, the last required column is the tens column and the digit after that is 5, so we would round up to 3,460.

(However, you should note that different rules may apply in certain situations - for example, in VAT calculations, the Customs and Excise rule is always to round down to the nearest one pence.)

(b) Significant Figures

By significant figures we mean all figures other than zeros at the beginning or end of a number. Thus:

- 632,000 has three significant figures as the zeros are not significant;
- 000,632 also has only three significant figures;
- 630,002, however, has six significant figures – the zeros here are significant as they occur in the middle of the number.

We can round off a number to a specified number of significant figures. This is likely to be done to aid the process of communication – i.e. to make the number language easier to understand – and is particularly important when we consider decimals later in the unit.

Consider the figure 2,017. This has four significant figures. To write it correct to three significant figures, we need to drop the fourth figure – i.e. the 7. We do this by using the rounding off rule – round down from 4 or round up from 5 – so 2,017 correct to three significant figures becomes 2,020. To write 2,017 correct to one significant figure, we consider the second significant figure, the zero, and apply the rule for rounding off, giving 2,000. Do not forget to include the insignificant zeros, so that the digits keep their correct ***place*** values.

Integers

These are simply whole numbers. Thus, 0, 1, 2, 3, 4, etc, are integers, as are all the negative whole numbers.

The following are not integers:

¾, 1½, 7¼, 0.45, 1.8, 20.25

By adjusting any number to the nearest whole number, we obtain an integer. Thus, a number which is not an integer can be ***approximated*** to become one by rounding off.

Practice Questions 1

1. Round off the following the figures to the nearest thousand:

   (a) 3,746

   (b) 8,231

   (c) 6,501

2. Round off the following figures to the nearest million:

   (a) 6,570,820

   (b) 8,480,290

   (c) 5,236,000

3. Correct the following to three significant figures:

   (a) 2,836

   (b) 3,894

   (c) 7,886

   (d) 9,726

4. How many significant figures are in the following?

   (a) 1,736

   (b) 22,300

   (c) 1,901

   (d) 26,000

Now check your answers with the ones given at the end of the unit.

C. ARITHMETIC

We will now revise some basic arithmetic processes. These may well be familiar to you, and although in practice it is likely you will perform many arithmetic operations using a calculator, it is important that you know how to do the calculations manually and understand all the rules that apply and the terminology used.

When carrying out arithmetic operations, remember that the position of each digit in a number is important, so layout and neatness matter.

Addition (symbol +)

When adding numbers, it is important to add each column correctly. If the numbers are written underneath each other in the correct position, there is less chance of error.

So, to add 246, 5,322, 16, 3,100 and 43, the first task is to set the figures out in columns:

	246
	5,322
	16
	3,100
	43
Total	8,727

The figures in the right hand column (the units column) are added first, then those in the next column to the left and so on. Where the sum of the numbers in a column amount to a figure greater than nine (for example, in the above calculation, the sum of the units column comes to 17), only the last number is inserted into the total and the first number is carried forward to be added into the next column. Here, it is 1 or, more precisely, one ten which is carried forward into the "tens" column.

Where you have several columns of numbers to add (as may be the case with certain statistical or financial information), the columns can be cross-checked to produce check totals which verify the accuracy of the addition:

Columns

	A	B	C	D	Totals
	246	210	270	23	749
	211	203	109	400	923
	29	107	13	16	165
	300	99	7	3	409
	47	102	199	91	439
Totals	833	721	598	533	2,685

Rows

Note that each row is added up as well as each column, thus providing a full cross-check.

Subtraction (symbol –)

Again, it is important to set out the figures underneath each other correctly in columns and to work through the calculation, column by column, starting with the units column.

So, to subtract 21 from 36, the first task is to set the sum out:

	36
	−21
Total	15

As with addition, you must work column by column, starting from the left – the units column.

Now consider this calculation:

	34
	−18
Total	16

Where it is not possible to subtract one number from another (as in the units column where you cannot take 8 from 4), the procedure is to "borrow" one from the next column to the left and add it to the first number (the 4). The one that you borrow is, in fact, one ten – so adding that to the first number makes 14, and subtracting 8 from that allows you to put 6 into the total. Moving on to the next column, it is essential to put back the one borrowed and this is added to the number to be subtracted. Thus, the second column now becomes $3 - (1 + 1)$, or $3 - 2$, which leaves 1.

Multiplication (symbol ×)

The multiplication of single numbers is relatively easy – for example, $2 \times 4 = 8$. When larger numbers are involved, it is again very helpful to set out the calculation in columns:

$$\begin{array}{r} 246 \\ \times\, 23 \\ \hline \end{array}$$

In practice, it is likely that you would use a calculator for this type of "long multiplication", but we shall briefly review the principles of doing it manually.

Before doing so, though, we shall note the names given to the various elements involved in a multiplication operation:

- the ***multiplicand*** is the number to be multiplied – in this example, it is 246;
- the ***multiplier*** is the number by which the multiplicand is to be multiplied – here it is 23; and
- the answer to a multiplication operation is called the ***product*** – so to find the product of two or more numbers, you multiply them together.

Returning to our example of long multiplication, the first operation is to multiply the multiplicand by the last number of the multiplier (the number of units, which is 3) and enter the product as a sub-total – this shown here as 738. Then we multiply the multiplicand by the next number of the multiplier (the number of tens, which is 2). However, here we have to remember that we are multiplying by tens and so we need to add a zero into the units column of this sub-total before entering the product for this operation (492). Thus, the actual product of 246×20 is 4,920. Finally, the sub-totals are added to give the product for whole operation:

	246
	× 23
Sub-total (units)	738
Sub-total (tens)	4,920
Total product	5,658

Division (symbol ÷)

In division, the calculation may be set out in a number of ways:

$360 \div 24$, or 360/24, or most commonly $\frac{360}{24}$

Just as there were three names for the various elements involved in a multiplication operation, so there are three names or terms used in division:

- the ***dividend*** is the number to be divided – in this example, it is 360;
- the ***divisor*** is the number by which the dividend is to be divided – here it is 24; and
- the answer to a division operation is called the ***quotient*** – here the quotient for our example is 15.

Division involving a single number divisor is relatively easy – for example, $16 \div 2 = 8$. For larger numbers, the task becomes more complicated and you should use a calculator for all long division.

The Rule of Priority

Calculations are often more complicated than a simple list of numbers to be added and/or subtracted. They may involve a number of different operations – for example:

$$8 \times 6 - 3 + \frac{8}{2}$$

When faced with such a calculation, the rule of priority in arithmetical operations requires that:

division and multiplication are carried out before addition and subtraction.

Thus, in the above example, the procedure would be:

(a) first the multiplication and division: $8 \times 6 = 48$; $\frac{8}{2} = 4$

(b) then, putting the results of the first step into the calculation, we can do the addition and subtraction: $48 - 3 + 4 = 49$

Brackets

Brackets are used extensively in arithmetic, and we shall meet them again when we explore algebra.

Look at the above calculation again. It is not really clear whether we should:

(a) multiply the eight by six; or

(b) multiply the eight by "six minus three" (i.e. by three); or even

(c) multiply the eight by "six minus three plus four" (i.e. by seven).

We use brackets to separate particular elements in the calculation so that we know exactly which operations to perform on which numbers. Thus, we could have written the above calculation as follows:

$$(8 \times 6) - 3 + \frac{8}{2}$$

This makes it perfectly clear what numbers are to be multiplied.

Consider another example:

$$12 \div 3 + 3 \times 6 + 1$$

Again there are a number of ways in which this calculation may be interpreted, for example:

$$(12 \div 3) + (3 \times 6) + 1$$

$$12 \div (3 + 3) \times (6 + 1)$$

$$(12 \div 3) + 3 \times (6 + 1)$$

$$(12 \div [3 + 3]) \times 6 + 1$$

Note that we can put brackets within brackets to further separate elements within the calculation and clarify how it is to be performed.

The rule of priority explained above now needs to be slightly modified. It must be strictly followed except when brackets are included. In that case, the contents of the brackets are evaluated first. If there are brackets within brackets, then the innermost brackets are evaluated first.

For example:

$2 + 3 \times (6 - 8 \div 2)$

$= 2 + 3 \times (6 - 4)$ *(within the brackets, the division first)*

$= 2 + 3 \times 2$

$= 2 + 6$ *(multiplication before addition)*

$= 8$

Thus we can see that brackets are important in signifying priority as well as showing how to treat particular operations.

Practice Questions 2

1. Work out the answers to the following questions, using a calculator if necessary.

   (a) 937×61

   (b) $1{,}460 \times 159$

   (c) $59{,}976 \div 63$

   (d) $19{,}944 \div 36$

2. Add each of the four columns in the following table and check your answer.

	A	**B**	**C**	**D**
	322	418	42	23
	219	16	600	46
	169	368	191	328
	210	191	100	169
Totals	833	721	598	533

3. Work out the answers to the following questions.

   (a) $(6 + 3) \times 6 + 4$

   (b) $(6 + 3) \times (6 + 4)$

   (c) $10 \times (5 + 2) - (10 \div 5)$

   (d) $24 + 24 \div (2 \times 2)$

   (e) $6 \times (24 \div [4 \times 2])$

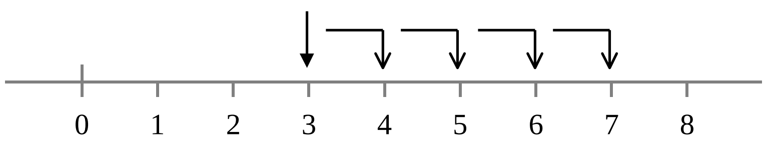
   (f) $(15 \div [3 + 2] + 6 + [4 \times 3] \times 2) \div 11$

Now check your answers with the ones given at the end of the unit.

D. DEALING WITH NEGATIVE NUMBERS

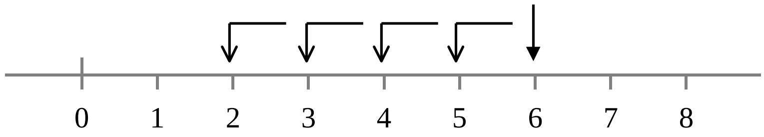
Addition and Subtraction

We can use a number scale, like the markings on ruler, as a simple picture of addition and subtraction. Thus, you can visualise the addition of, say, three and four by finding the position of three on the scale and counting off a further four divisions to the ***right***, as follows:



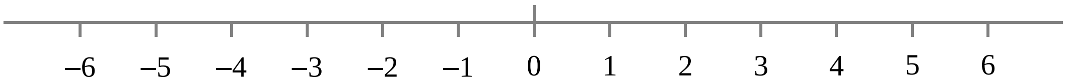
This brings us to the expected answer of seven.

Similarly we can view subtraction as moving to the ***left*** on the scale. Thus, 6 minus 4 is represented as follows:

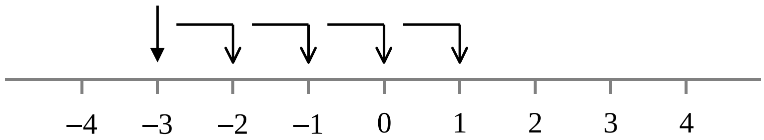


This illustrates the sum $6 - 4 = 2$.

Negative numbers can also be represented on the number scale. All we do is extend the scale to the left of zero:



We can now visualise the addition and subtraction of negative numbers in the same way as above. Consider the sum $(-3) + 4$. Start at –3 on the scale and count four divisions to the right:



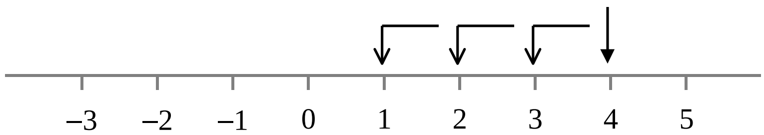
Therefore, $(-3) + 4 = 1$.

(Note that, here, we have shown the figure of "minus 3" in brackets to avoid confusion with a the addition and subtraction signs.)

The order of the figures of these calculations does not matter, so we can also write:

$$(-3) + 4 = 4 + (-3) = 1$$

We can illustrate the second sum as follows:



This is the same as $4 - 3$. In effect, therefore, a plus and a minus together make a minus.

With subtraction, the process is slightly different. To subtract a minus number, we change the subtraction sign to a plus and treat the number as positive. For example:

$$4 - (-3) = 4 + 3 = 7$$

To sum up: ***A plus and minus make a minus***

Two minuses make a plus

Consider the follow examples. (Draw your own number scales to help visualise the sums if you find it helpful.)

$7 - (-3) = 7 + 3 = 10$

$2 + (-6) = 2 - 6 = -4$

The rules for multiplication and division are as follows:

Figure 1.3: Multiplying and dividing using negative numbers

Type of calculation	Product/Quotient	Example
Minus times plus	minus	$-3 \times 7 = -21$
Minus times minus	plus	$-7 \times (-3) = 21$
Minus divide by plus	minus	$-21 \div 3 = -7$
Minus divide by minus	plus	$-21 \div (-3) = 7$
Plus divide by minus	minus	$21 \div (-3) = -7$

Practice Questions 3

Solve the following sums, using a number scale if necessary:

1. $7 + (-4)$
2. $(-4) + 7$
3. $8 - (-6)$
4. $6 + (-8)$
5. $(-9) + 3$
6. $(-8) - (-2)$
7. $(-5) + (-4)$
8. $(-3) \times (-5)$
9. $(-7) \div (-7)$
10. $10 \times (-10)$

Now check your answers with the ones given at the end of the unit.

E. FRACTIONS

A fraction is a part of whole number.

When two or more whole numbers are multiplied together, the product is always another whole number. In contrast, the division of one whole number by another does not always result in a whole number. Consider the following two examples:

$12 \div 5 = 2$, with a remainder of 2

$10 \div 3 = 3$, with a remainder of 1

The remainder is not a whole number, but a part of a whole number – so, in each case, the result of the division is a whole number and a fraction of a whole number. The fraction is expressed as the remainder divided by the original divisor and the way in which it is shown, then, is in the same form as a division term:

$12 \div 5 = 2$, with a remainder of 2 parts of 5, or two divided by five, or $\frac{2}{5}$

$10 \div 3 = 3$, with a remainder of 1 parts of 3, or one divided by three, or $\frac{1}{3}$

Each part of the fraction has a specific terminology:

- the top number is the ***numerator***; and
- the bottom number is the ***denominator***.

Thus, for the above fractions, the numerators are 2 and 1, and the denominators are 5 and 3.

When two numbers are divided, there are three possible outcomes.

(a) The numerator is larger than the denominator

In this case, the fraction is known as an ***improper fraction***. The result of dividing out an improper fraction is a whole number plus a part of one whole, as in the two examples above. Taking two more examples:

$\frac{42}{9}$ = 4 whole parts and a remainder of 6 parts of 9, or $4\frac{6}{9}$

$\frac{30}{24}$ = 1 whole part and a remainder of 6 parts of 30, or $1\frac{6}{24}$

(b) The denominator is larger than the numerator

In this case, the fraction is known as a ***proper fraction***. The result of dividing out a proper fraction is only a part of one whole and, therefore, a proper fraction has a value of less than one.

Examples of proper fractions are:

$\frac{1}{5}, \frac{2}{3}, \frac{4}{7}, \frac{4}{5}$, etc.

(2) The numerator is the same as the denominator

In this case, the expression is not a true fraction. The result of dividing out is one whole part and no remainder. The answer must be 1. For example:

$$\frac{5}{5} = 5 \div 5 = 1$$

Fractions are often thought of as being quite difficult. However, they are not really hard as long as you learn the basic rules about how they can be manipulated. We shall start by looking at these rules before going on to examine their application in performing arithmetic operations on fractions.

Basic Rules for Fractions

- ***Cancelling down***

  Both the numerator and the denominator in a fraction can be any number. The following are all valid fractions:

  $\frac{2}{5}, \frac{20}{50}, \frac{47}{93}, \frac{6}{9}$

The rule in expressing fractions, though, is always to reduce the fraction to the ***smallest possible whole numbers*** for ***both*** its ***numerator*** and ***denominator***. The resulting fraction is then in its ***lowest possible terms***.

The process of reducing a fraction to its lowest possible terms is called ***cancelling down***. To do this, we follow another simple rule. Divide both the numerator and denominator by the largest whole number which will divide into them exactly.

Consider the following fraction from those listed previously:

$$\frac{20}{50}$$

Which whole number will divide into both 20 and 50? There are several: 2, 5, 10. The rule, though, states that we use the largest whole number, so we should use 10. Dividing both parts of the fraction by 10 reduces it to:

$$\frac{2}{5}$$

Note that we have not changed the value of the fraction, just the way in which the parts of the whole are expressed. Thus:

$$\frac{20}{50} = \frac{2}{5}$$

A fraction which has ***not*** been reduced to its lowest possible terms is called a ***vulgar fraction***.

Cancelling down can be tricky process since it is not always clear what the largest whole number to use might be. For example, consider the following fraction:

$$\frac{165}{231}$$

In fact, both terms can be divided by 33, but you are very unlikely to see that straight away. However, you would probably quickly see that both are divisible by 3, so you could start the process of cancelling down by dividing by three:

$$\frac{165 \div 3}{231 \div 3} = \frac{55}{77}$$

This gives us a new form of the same fraction, and we can now see that both terms can be further divided – this time by 11:

$$\frac{55 \div 11}{77 \div 11} = \frac{5}{7}$$

So, large vulgar fractions can often be reduced in stages to their lowest possible terms.

- ***Changing the denominator to required number***

The above process works in reverse. If we wanted to express a given fraction in a different way, we can do that by multiplying both the numerator and denominator by the same number. So, for example, $\frac{2}{5}$ could be expressed as $\frac{8}{20}$ by multiplying both terms by 4.

If we wanted to change the denominator to a specific number – for example, to express ½ in eighths – the rule is to divide the required denominator by the existing denominator (8 ÷ 2 = 4) and then multiply both terms of the original fraction by this result:

$$\frac{1 \times 4}{2 \times 4} = \frac{4}{8}$$

Note, again, that we have not changed the value of the fraction, just the way in which the parts of the whole are expressed.

- ***Changing an improper fraction into a whole or mixed number***

  Suppose we are given the following improper fraction:

  $$\frac{35}{8}$$

  The rule for converting this is to divide the numerator by the denominator and place any remainder over the denominator. Those we find out how many whole parts there are and how many parts of the whole (in this case, eighths) are left over:

  $$\frac{35}{8} \;=\; 35 \div 8 \;=\; 4 \text{ and a remainder of three } \;=\; 4\frac{3}{8}$$

  This result is known as a ***mixed number*** – one comprising a whole number and a fraction.

- ***Changing a mixed number into an improper fraction***

  This is the exact reverse of the above operation. The rule is to multiply the whole number by the denominator in the fraction, then add the numerator of the fraction to this product and place the sum over the denominator.

  For example, to write $3\frac{2}{3}$ as an improper fraction:

  (a) first multiply the whole number (3) by the denominator (3) $= 3 \times 3 = 9$, changing the three whole ones into nine thirds;

  (b) then add the numerator (2) to this $= 2 + 9 = 1$, thus adding the two thirds to the nine thirds from the first operation;

  (c) finally place the sum over the denominator $= \frac{11}{3}$

Practice Questions 4

1. Reduce the following fractions to their lowest possible terms:

   (a) $\frac{9}{36}$ (b) $\frac{8}{64}$

   (c) $\frac{21}{70}$ (d) $\frac{18}{54}$

   (e) $\frac{14}{49}$ (f) $\frac{25}{150}$

   (g) $\frac{36}{81}$ (h) $\frac{45}{135}$

   (i) $\frac{24}{192}$ (j) $\frac{33}{88}$

2. Change the following fractions as indicated:

   (a) $\frac{1}{5}$ to 10ths (b) $\frac{3}{5}$ to 20ths

   (c) $\frac{4}{9}$ to 81sts (d) $\frac{9}{11}$ to 88ths

3. Change the following improper fractions to mixed numbers:

(a) $\frac{12}{5}$ (b) $\frac{17}{8}$

(c) $\frac{25}{6}$ (d) $\frac{31}{3}$

4. Change the following mixed numbers to improper fractions:

(a) $2\frac{2}{3}$ (b) $3\frac{3}{8}$

(c) $4\frac{1}{8}$ (d) $9\frac{3}{4}$

Now check your answers with the ones given at the end of the unit.

Adding and Subtracting with Fractions

(a) Proper fractions

Adding or subtracting fractions depends upon them having the same denominator. It is not possible to do these operations if the denominators are different.

Where the denominators are the same, adding or subtracting is just a simple case of applying the operation to numerators. For example:

$$\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$$

$$\frac{5}{8} - \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Where the denominators are not the same, we need to change the form of expression so that they become the same. This process is called finding the common denominator.

Consider the following calculation:

$$\frac{2}{5} + \frac{3}{10}$$

We cannot add these two fractions together as they stand – we have to find the common denominator. In this case, we can change the expression of the first fraction by multiplying by both terms by 2, thus making its denominator 10 – the same as the second fraction:

$$\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$$

Not all sums are as simple as this. It is often the case that we have to change the form of expression of all the fractions in a sum in order to give them all the same denominator.

Consider the following calculation:

$$\frac{3}{5} + \frac{1}{4} - \frac{1}{10}$$

The lowest common denominator for these three fractions is 20. We must then convert all three to the new denominator by applying the rule explained above for changing a fraction to a required denominator:

$$\frac{3}{5} + \frac{1}{4} - \frac{1}{10} = \frac{12}{20} + \frac{5}{20} - \frac{2}{20} = \frac{15}{20} = \frac{3}{4}$$

The rule is simple:

- find the lowest common denominator – and where this is not readily apparent, a common denominator may be found by multiplying the denominators together;
- change all the fractions in the calculation to this common denominator;
- then add (or subtract) the numerators and place the result over the common denominator.

(b) Mixed numbers

When adding mixed numbers, we deal with the integers and the fractions separately. The procedure is as follows:

- first add all the integers together;
- then add the fractions together as explained above; and then
- add together the sum of the integers and the fractions.

Consider the following calculation:

$$3 + 4\frac{1}{4} + 5\frac{1}{2} + 2\frac{3}{4}$$

$$= (3 + 4 + 5 + 2) + \left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right)$$

$$= 14 + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right)$$

$$= 14 + \frac{6}{4}$$

$$= 14 + 1\frac{1}{2}$$

$$= 15\frac{1}{2}$$

In general, the same principles are applied to subtracting mixed numbers. For example:

$$4\frac{5}{6} - 1\frac{1}{2} = (4 - 1) + \left(\frac{5}{6} - \frac{1}{2}\right) = 3 + \left(\frac{5}{6} - \frac{3}{6}\right) = 3 + \frac{2}{6} = 3\frac{1}{3}$$

However, the procedure is slightly different where the second fraction is larger than the first. Consider this calculation:

$$4\frac{1}{2} - 1\frac{5}{6}$$

Here, it is not possible to subtract the fractions even after changing them to the common denominator:

$$\frac{3}{6} - \frac{5}{6}$$

To complete the calculation we need to take one whole number from the integer in the first mixed number, convert it to a fraction with the same common denominator and add it to the fraction in the mixed number. Thus:

$$4\frac{1}{2} \rightarrow 3 + \left(1 + \frac{1}{2}\right) = 3 + \left(\frac{6}{6} + \frac{3}{6}\right) = 3\frac{9}{6}$$

We can now carry out the subtraction as above:

$$4\frac{1}{2} - 1\frac{5}{6} = 3\frac{9}{6} - 1\frac{5}{6} = (3-1) + \left(\frac{9}{6} - \frac{5}{6}\right) = 2 + \frac{4}{6} = 2\frac{2}{3}$$

Practice Questions 5

Carry out the following calculations:

1. $\frac{1}{3} + \frac{1}{4}$
2. $\frac{1}{6} + \frac{1}{10}$
3. $\frac{1}{8} + \frac{2}{3}$
4. $\frac{1}{5} + \frac{1}{6}$
5. $\frac{2}{5} + \frac{3}{4}$
6. $\frac{1}{3} + \frac{3}{8}$
7. $\frac{3}{4} - \frac{1}{8}$
8. $\frac{2}{3} - \frac{1}{4}$
9. $\frac{5}{6} - \frac{1}{3}$
10. $\frac{7}{8} - \frac{1}{5}$
11. $\frac{5}{7} - \frac{1}{3}$
12. $\frac{3}{4} - \frac{3}{8}$
13. $2\frac{1}{2} + 1\frac{3}{4}$
14. $4\frac{3}{8} + 2\frac{3}{4} + 1\frac{1}{2}$
15. $\frac{2}{3} + 3 + 1\frac{1}{2} + 2\frac{1}{5}$
16. $1\frac{3}{4} - \frac{2}{3}$
17. $4\frac{5}{9} - 2\frac{1}{3}$
18. $6\frac{1}{8} - 2\frac{1}{3}$
19. $2\frac{1}{8} + 2\frac{1}{3} - 2\frac{1}{6}$
20. $4\frac{3}{4} + 1\frac{2}{3} - 2\frac{1}{2}$

Now check your answers with the ones given at the end of the unit.

Multiplying and Dividing Fractions

(a) Multiplying a fraction by a whole number

In this case, we simply multiply the ***numerator*** by the whole number and place the product (which is now the new numerator) over the ***old*** denominator. For example:

$$\frac{2}{3} \times 5 = \frac{2 \times 5}{3} = \frac{10}{3} = 3\frac{1}{3}$$

$$\frac{3}{4} \times 7 = \frac{3\times 7}{4} = \frac{21}{4} = 5\frac{1}{4}$$

We often see the word "of" in a calculation involving fractions – for example, ¼ of £12,000. This is just another way of expressing multiplication.

(b) Dividing a fraction by a whole number

In this case, we simply multiply the ***denominator*** by the whole number and place the ***old*** numerator over the product (which is now the new denominator). For example:

$$\frac{3}{4} \div 5 = \frac{3}{4\times 5} = \frac{3}{20}$$

$$\frac{2}{3} \div 5 = \frac{2}{3\times 5} = \frac{2}{15}$$

(c) Multiplying one fraction by another

In this case, we multiply the numerators by each other and the denominators by each other, and then reduce the result to the lowest possible terms. For example:

$$\frac{3}{4} \times \frac{2}{3} = \frac{3\times 2}{4\times 3} = \frac{6}{12} = \frac{1}{2}$$

$$\frac{3}{8} \times \frac{3}{5} = \frac{3\times 3}{8\times 5} = \frac{9}{40}$$

(d) Dividing by a fraction

In this case, we turn the divisor upside down and ***multiply***. For example:

$$4 \div \frac{3}{4} = 4 \times \frac{4}{3} = \frac{16}{3} = 5\frac{1}{3}$$

$$\frac{1}{6} \div \frac{1}{3} = \frac{1}{6} \times \frac{3}{1} = \frac{3}{6} = \frac{1}{2}$$

(e) Dealing with mixed numbers

All that we have said so far in this section applies to the multiplication and division of proper and improper fractions. Before we can apply the same methods to mixed numbers, we must complete one further step – convert the mixed number to an improper fraction. For example:

$$5\frac{1}{2} \times \frac{4}{11} = \frac{11}{2} \times \frac{4}{11} = \frac{44}{22} = 2$$

$$4\frac{3}{4} \div \frac{1}{2} = \frac{19}{4} \times \frac{2}{1} = \frac{38}{4} = 9\frac{1}{2}$$

(f) Cancelling during multiplication

We have seen how it is possible to reduce fractions to their lowest possible terms by cancelling down. This involves dividing both the numerator and the denominator by the same number, for example:

$$\frac{6}{8} \text{ cancels down to } \frac{3}{4} \text{ by dividing both terms by 2}$$

We often show the process of cancelling down during a calculation by crossing out the cancelled terms and inserting the reduced terms as follows:

$$\frac{\cancel{8}\ 2}{\cancel{12}\ 3} = \frac{2}{3}$$

(It is always helpful to set out all the workings throughout a calculation. Some calculations can be very long, involving many steps. If you make a mistake somewhere, it is much easier to identify the error if all the workings are shown. It is also the case that, in an examination, examiners will give "marks for workings", even if you make a mistake in the arithmetic. However, if you do not show all the workings, they will not be able to follow the process and see that the error is purely mathematical, rather than in the steps you have gone through.)

Cancellation can also be used in the multiplication of fractions. In this case it can be applied across the multiplication sign – dividing the numerator of one fraction and the denominator of the other by the same number. For example:

$$\frac{\cancel{3}\,1}{4} \times \frac{1}{\cancel{12}\,4} = \frac{1}{16}$$

Remember that every cancelling step must involve a numerator and a denominator. Thus, the following operation would be wrong:

$$\frac{3}{\cancel{4}\,1} \times \frac{1}{\cancel{12}\,3} = \frac{3}{3}$$

Note, too, that you ***cannot*** cancel across addition and subtraction signs.

However, you ***can*** use cancellation during a division calculation at the step when, having turned the divisor upside down, you are multiplying the two fractions. So, repeating the calculation involving mixed numbers from above, we could show it as follows:

$$4\frac{3}{4} \div \frac{1}{2} = \frac{19}{\cancel{4}\,2} \times \frac{\cancel{2}\,1}{1} = \frac{19}{2} = 9\frac{1}{2}$$

Practice Questions 6

Carry out the following calculations:

1. $7 \times \frac{3}{4}$
2. $8 \times \frac{2}{3}$
3. $5 \times \frac{3}{6}$
4. $6 \times \frac{3}{7}$
5. $\frac{2}{3} \div 6$
6. $\frac{3}{4} \div 6$
7. $\frac{7}{8} \div 3$
8. $\frac{5}{7} \div 4$

Now check your answers with the ones given at the end of the unit.

F. DECIMALS

Decimals are an alternative way of expressing a particular part of a whole. The term "decimal" means "in relation to ten", and decimals are effectively fractions expressed in tenths, hundredths, thousandths, etc.

Unlike fractions, though, decimals are written completely differently and this makes carrying out arithmetic operations on them much easier.

Decimals do not have a numerator or a denominator. Rather, the part of the whole is shown by a number following a ***decimal point***:

- the first number following the decimal point represents the number of tenths, so 0.1 is one tenth part of a whole and 0.3 is three tenths of a whole;
- the next number to the right, if there is one, represents the number of hundredths, so 0.02 is two hundredths of a whole and 0.67 is six tenths and seven hundredths (i.e. 67 hundredths) of a whole;
- the next number to the right, again if there is one, represents the number of thousandths, so 0.005 is five thousandths of a whole and 0.134 is 134 thousandths of a whole;

and so on up to as many figures as are necessary.

We can see the relationship between decimals and fractions as follows:

$$0.1 = \frac{1}{10}, \ 0.3 = \frac{3}{10}, \ 1.6 = 1\frac{6}{10}$$

$$0.01 = \frac{1}{100}, \ 0.67 = \frac{67}{100}, \ 3.81 = 3\frac{81}{100}$$

$$0.001 = \frac{1}{1000}, \ 0.054 = \frac{54}{1000}, \ 5.382 = 5\frac{382}{1000}$$

We can see that any fraction with a denominator of ten, one hundred, one thousand, ten thousand, etc. can be easily converted into a decimal. However, fractions with a different denominator are also relatively easy to convert by simple division.

Arithmetic with Decimals

Performing addition, subtraction, multiplication and division with decimals is exactly the same as carrying out those same operations on whole numbers. The one key point to bear in mind is to get the position of the decimal point in the right place. Layout is all important in achieving this.

(a) Adding and subtracting

To add or subtract decimals, arrange the numbers in columns with the decimal points all being in the same vertical line. Then you can carry out the arithmetic in the same way as for whole numbers.

For example, add the following: 3.6 + 7.84 + 9.172

The layout for this is as follows:

	3.6
	7.84
	9.172
Total	20.612

Similarly, subtraction is easy provided that you follow the same rule of ensuring that the decimal points are aligned vertically. For example, subtract 18.857 from 63.7:

	63.7
	–18.857
Total	44.843

(b) Multiplying decimals

It is easier to use a calculator to multiply decimals, although you should ensure that you take care to get the decimal point in the correct place when both inputting the figures and when reading the answer.

The principles of multiplying decimals manually are the same as those considered earlier in the unit for whole numbers. However, it is important to get the decimal point in the right place.

There is a simple rule for this – the answer should have the same number of decimal places as there were in the question.

Consider the sum 40×0.5. The simplest way of carrying this out is to ignore the decimal points in the first instance, multiply the sum out as if the figures were whole numbers and then put back the decimal point in the answer. The trick is to put the decimal point back in the right place!

So, ignoring the decimal points, we can multiply the numbers as follows:

$$40 \times 5 = 200$$

Now we need to put the decimal point into the answer. To get it in the right place, we count the number of figures after a decimal point in the original sum – there is just one, in the multiplier (0.5) – and place the decimal point that number of places from the right in the answer:

20.0 or just 20

If the sum had been 0.4×0.5, what would the answer be?

The multiplication would be the same, but this time there are 2 figures after decimal points in the original sum. So, we insert the decimal point two places to the right in the answer:

0.2

One final point to note about multiplying decimals is that, ***to multiply a decimal by ten***, you simply ***move the decimal point one place to the right***. For example:

$$3.62 \times 10 = 36.2$$

Likewise, to multiply by 100, you move the decimal point ***two*** places to the right:

$$3.62 \times 100 = 362$$

This can be a very useful technique, enabling quick manual calculations to be done with ease.

(c) Division with decimals

Again, it is far easier to use a calculator when dividing with decimals.

However, when dividing by ten or by one hundred (or even by a thousand or a million), we can use the technique noted above in respect of multiplication to find the answer quickly without using a calculator:

- ***to divide by ten***, simply ***move the decimal point one place to the left*** – for example, $68.32 \div 10 = 1.832$;
- to divide by one hundred, simply move the decimal point ***two*** places to the left – for example, $68.32 \div 100 = 0.6832$.

(d) Converting fractions to decimals, and vice versa

Fractions other than tenths, hundredths, thousandths, etc. can also be converted into decimals. It is simply a process of dividing the numerator by the denominator and expressing the answer in decimal form. You can do this manually, but in most instances it is easier to use a calculator.

For example, to change $\frac{3}{5}$ into decimal form, you divide 3 by 5 (i.e. 3 ÷ 5).

Using a calculator the answer is easily found: 0.6

However, not all fractions will convert so easily into decimals in this way.

Consider the fraction $\frac{1}{3}$.

Dividing this out (or using a calculator) gives the answer 0.3333333, or however many 3s for which there is room on the paper or the calculator screen. This type of decimal, where one or more numbers repeat infinitely, is known as a recurring decimal.

In most business applications, we do not need such a degree of precision, so we abbreviate the decimal to a certain size – as we shall discuss in the next section.

To change a decimal into a fraction, we simply place the decimal over ten or one hundred, etc, depending upon the size of the decimal. Thus, the decimal becomes the numerator and the denominator is 10 or 100, etc. For example:

$$0.36 = \frac{36}{100} = \frac{9}{25}$$

$$3.736 = 3\frac{736}{1000} = 3\frac{92}{125}$$

Limiting the number of decimal places

We noted above that certain fractions do not convert into exact decimals. The most common examples are one third and two thirds, which convert into 0.3333333, etc. and 0.6666666, etc, respectively. (When referring to these types of recurring decimals, we would normally just quote the decimal as, say, "0.3 recurring".) However, there are also plenty of other fractions which do convert into exact decimals, but only at the level of thousandths or greater. For example:

$$\frac{2}{7} = 0.2857142, \text{ or } \frac{11}{17} = 0.6470568$$

This gives us a problem: how much detail do we want to know? It is very rare that, in business applications, we would want to know an answer precise to the millionth part of a whole, although in engineering, that level of precision may be critical. Such large decimals also present problems in carrying out arithmetic procedures – for example, it is not easy to add the above two decimals together and multiplying them does not bear thinking about.

In most business circumstances, then, we limit the number of decimal places to a manageable number. This is usually set at two or three decimal places, so we are only interested in the level of detail down to hundredths or thousandths. We then speak of a number or an answer as being "***correct to two (or three) decimal places***".

For decimals with a larger number of decimal places than the required number, we use the principles of rounding discussed earlier in the unit to reduce them to the correct size. Thus, we look at the digit in the decimal ***after*** the one in the last required place and then:

- for figures of five or more, round up the figure in the last required place;
- for figures of four or less, round down by leaving the figure in the last required place unchanged.

So, if we want to express the decimal form of two sevenths (0.2857142) correct to three decimal places, we look at the digit in the fourth decimal place: 7. Following the rule for rounding, we would then round up the number in the third decimal place to give us 0.286.

Now consider the decimal forms of one third and two thirds. If we were to express these correct to two decimal places, these would be 0.33 and 0.67 respectively.

Working to only two or three decimal places greatly simplifies decimals, but you should note that the level of accuracy suffers. For example, multiplying 0.33 by two gives us 0.66, but we have shown two thirds as being 0.67. These types of small error can increase when adding, subtracting, multiplying or dividing several decimals which have been rounded – something which we shall see later in the course in respect of a set of figures which should add up to 100, but do not. In such cases, we need to draw attention to the problem by stating that "errors are due to rounding".

Practice Questions 7

1. Carry out the following calculations:

(a) $3.67 + 4.81 + 3.2$ (b) $8.36 + 4.79 + 6.32$

(c) $7.07 + 8.08 + 11.263$ (d) $11.14 + 8.16 + 3.792$

(e) $8.84 - 3.42$ (f) $7.36 - 2.182$

(g) $10.8 - 4.209$ (h) $12.6 - 3.43$

(i) 3.62×2.8 (j) 4.8×3.3

(k) 5.8×4.1 (l) 0.63×5.2

(m) $154 \div 2.2$ (n) $2.72 \div 1.7$

(o) $11.592 \div 2.07$ (p) $2.6 \div 0.052$

2. Change the following fractions into decimal form, correct to the number of decimal places specified:

(a) $\frac{2}{5}$ to 2 dp (b) $\frac{3}{8}$ to 3 dp

(c) $\frac{3}{4}$ to 2 dp (d) $\frac{7}{8}$ to 3 dp

(e) $\frac{5}{8}$ to 3 dp (f) $\frac{5}{6}$ to 3 dp

(g) $\frac{4}{7}$ to 3 dp (h) $\frac{5}{16}$ to 3 dp

3. Change the following decimals into fractions and reduce them to their lowest possible terms

(a) 0.125 (b) 0.60

(c) 1.75 (d) 0.750

Now check your answers with the ones given at the end of the unit.

G. PERCENTAGES

A percentage is a means of expressing a fractions in parts of a hundred – the words "per cent" simply mean "per hundred", so when we say "percentage", we mean "out of a hundred". The expression "20 per cent" means 20 out of 100. The per cent sign is %, so 20 per cent is written as 20%.

- 4% means four hundredths of a whole, or four parts of a hundred $= \frac{4}{100}$
- 28% means twenty-eight hundredths of a whole, or twenty-eight parts of a hundred $= \frac{28}{100}$
- 100% means one hundred hundredths of a whole, or a hundred parts of a hundred $= \frac{100}{100} = 1$

Percentages are used extensively in many aspects of business since they are generally more convenient and straightforward to use than fractions and decimals. It is important, therefore, that you are fully conversant with working with them.

Arithmetic with Percentages

Multiplication and division have no meaning when applied to percentages, so we need only be concerned with their addition and subtraction. These are quite straightforward, but you need to remember that we can only add or subtract percentages if they are parts of the same whole.

Take an example: If Peter spends 40% of his income on rent and 25% on household expenses, what percentage of his income remains? Here both percentages are parts of the same whole – Peter's income – so we can calculate the percentage remaining as follows:

$$100\% - (40\% + 25\%) = 35\%$$

However, consider this example. If Alan eats 20% of his cake and 10% of his pie, what percentage remains? Here the two percentages are not parts of the same whole – one refers to cake and the other to pie. Therefore, it is not possible to add the two together.

Percentages and Fractions

Since a percentage is a part of one hundred, to change a percentage to a fraction you simply divide it by 100. In effect all this means is that you place the percentage over 100 and then reduce it to its lowest possible terms.

For example:

$$60\% = \frac{60}{100} = \frac{2}{5}$$

$$24\% = \frac{24}{100} = \frac{6}{25}$$

Changing a fraction into a percentage is the reverse of this process – simply multiply by 100. For example:

$$\frac{3}{8} \times 100 = \frac{300}{8}\% = 37\tfrac{1}{2}\% \text{ or } 37.5\%$$

$$\frac{2}{3} \times 100 = \frac{200}{3}\% = 66.67\%$$

As you become familiar with percentages you will get to know certain equivalent fractions well. Here are some of them:

$12\tfrac{1}{2}\% = \frac{1}{8}$ $\quad$ $20\% = \frac{1}{5}$

$25\% = \frac{1}{4}$ $\quad$ $33.3\% = \frac{1}{3}$

$50\% = \frac{1}{2}$ $\quad$ $75\% = \frac{3}{4}$

Percentages and Decimals

To change a percentage to a decimal, again you divide it by 100. As we saw in the last section, this is easy – you simply move the decimal point two places to the left. Thus:

$$11.5\% = 0.115$$

Working the conversion the other way is equally easy. To change a decimal to a percentage, move the decimal point two places to the right:

$$0.36\% = 36\%$$

Calculating Percentages

To find a particular percentage of a given number, change the percentage rate into a fraction or a decimal and then multiply the given number by that fraction/decimal.

For example, to find 15% of £260:

$$£260 \times \frac{15}{100} = \frac{£3900}{100} = £39$$

or $\quad £260 \times 0.15 = £39$

To express one number as a percentage of another, we convert the numbers into a fraction with the first number as the numerator and the second the denominator, and then change the fraction into a percentage by multiplying by 100.

For example, what percentage of cars have been sold if 27 have gone from a total stock of 300?

$$\frac{27}{300} \times 100 = \frac{2700}{300} = 9\%$$

Do not forget that percentages refer to parts of the same whole and, therefore, can only be only be calculated where the units of measurement of the quantities concerned are the same.

For example, to show 23 pence as a percentage of £2, we need to express both quantities in the same units before working out the percentage. This could be both as pence or both as pounds:

$$\frac{23}{200} \times 100 = 11.5\%$$

or $$\frac{0.23}{2} \times 100 = 11.5\%$$

There are occasions when we deal with percentages greater than 100%.

For example, if a man sells a car for £1,200 and says that he made 20% profit, how much did he buy it for?

The cost plus the profit = 120% of cost = £1,200

The cost, therefore, is 100%

To work out the cost, we need to multiply the selling price by $\frac{100}{120}$:

$$\text{Cost} = 1{,}200 \times \frac{100}{120} = £1{,}000$$

To prove that this is correct, we work the calculation through the other way. The cost was £1,000 and he sold it at 20% profit, so the selling price is:

$$£1{,}000 + 20\% \text{ of } £1{,}000 = £(1{,}000 + 0.2 \times 1{,}000) = £1{,}200, \text{ as given above.}$$

Practice Questions 8

1. Change the following fractions into percentages:

   (a) $\frac{3}{100}$ (b) $\frac{3}{8}$

2. Change the following percentages into fractions and reduce them to their lowest possible terms

   (a) 2% (b) 8%

   (c) 7½% (d) 18%

3. Calculate the following:

   (a) 15% of £200 (b) 25% of 360°

   (c) 12½% of 280 bottles (d) 27% of £1,790

   (e) 17.5% of £138

4. Calculate the following:

   (a) 60 as a percentage of 300 (b) 25 as a percentage of 75

   (c) £25 as a percentage of £1,000 (d) 30 as a percentage of 20

Now check your answers with the ones given at the end of the unit.

H. RATIOS

A ratio is a way of expressing the relationship between two quantities. It is essential that the two quantities are of expressed in the same units of measurement – for example, pence, number of people, etc. – or the comparison will not be valid.

For example, if we wanted to give the ratio of the width of a table to its depth where the width is 2 m and the depth is 87 cm, we would have to convert the two quantities to the same units before giving the ratio. Expressing both in centimetres would give the ratio of 200 to 87.

Note that the ratio itself is not in any particular unit – it just shows the relationship between quantities of the same unit.

We use a special symbol (the colon symbol :) in expressing ratios, so the correct form of showing the above ratio would be:

200 : 87 (pronounced "200 to 87")

There are two general rules in respect of ratios.

- They should always be reduced to their lowest possible terms (as with vulgar fractions). To do this, express the ratio as a fraction by putting first figure over the second and cancel the resulting fraction. Then re-express the ratio in the form of numerator : denominator.

  For example, consider the ratio of girls to boys in a group of 85 girls and 17 boys. The ratio would be 85 : 17. This can be reduced as follows:

  $$\frac{85}{17} = \frac{1}{5}$$

  The ratio of 85 to 17 is, therefore, 5 : 1.

- Ratios should always be expressed in whole numbers. There should not be any fractions or decimals in them.

  Consider the ratio of average miles per gallon for cars with petrol engines to that for cars with diesel engines where the respective figures are 33½ mpg and 47 mpg. We would not express this as 33½ : 47, but convert the figures to whole numbers by, here, multiplying by 2 to give:

  67 : 94

Ratios are used extensively in business to provide information about the way in which one element relates to another. For example, a common way of analysing business performance is to compare profits with sales. If we look at the ratio of profits to sales across two years, we may be able to see if this aspect of business performance is improving , staying the same or deteriorating. (This will be examined elsewhere in you studies.)

Another application is to work out quantities according to a particular ratio. For example, ratios are often used to express the relationship between partners in a partnership – sharing profits on a basis of, say, 50 : 50 or 80 : 20.

Consider the case of three partners – Ansell, Boddington and Devenish – who share the profits of their partnership in the ratio of 3 : 1 : 5. If the profit for a year is £18,000, how much does each partner receive?

To divide a quantity according to a given ratio:

- first add the terms of the ratio to find the total number of parts;
- then find what fraction each term of the ratio is to the whole; and
- finally divide the total quantity into parts according to the fractions.

For the Ansell, Boddington and Devenish partnership, this would be calculated as follows:

The profit is divided into 3 + 1 + 5 = 9 parts.

Therefore:

Ansell receives $\frac{3}{9}$ of the profits;

Boddington receives $\frac{1}{9}$ of the profits; and

Devenish receives $\frac{5}{9}$ of the profits.

Therefore:

Ansell gets $\frac{3}{9} \times £18,000 = £6,000$

Boddington gets $\frac{1}{9} \times £18,000 = £2,000$

Devenish gets $\frac{5}{9} \times £18,000 = £10,000$

Practice Questions 9

1. Express the following sales and profits figures as profit to sales ratios for each firm:

Firm	Sales	Profits
	£	£
A	100,000	25,000
B	63,000	7,000
C	30,000	10,000
D	75,000	15,000

2. Calculate the distribution of profits for the following partnerships:

   (a) X, Y and Z share profits in the ratio of 3 : 4: 5. Total profit is £60,000

   (b) F, G and H share profits in the ratio of 2 : 3 : 3. Total profit is £7,200

Now check your answers with the ones given at the end of the unit.

I. FURTHER KEY CONCEPTS

In this last section we shall briefly introduce a number of concepts which will be used extensively later in the course. Here we concentrate mostly on what these concepts mean, and you will get the opportunity to practice their use in later units. However, we do include practice in relation to the expression of numbers in standard form.

Indices

Indices are found when we multiply a number by itself one or more times. The number of times that the multiplication is repeated is indicated by a superscript number to the right of the number being multiplied. Thus:

2×2	is written as 2^2 (the little "2" raised up is the superscript number)
$2 \times 2 \times 2$	is written as 2^3
$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	is written as 2^7
etc.	

The number being multiplied (here, 2) is termed the ***base*** and the superscript number is called the ***exponent*** or ***index*** of its power. Thus, in the case of, say, 2^6 we refer to the term as "two raised to the sixth power" or "two to the power of six". (Strictly, 2^2 should be referred to as "two to the power of two", but we normally call it "2 squared" and, similarly, 2^3 is referred to as "2 cubed".)

The ***power*** of a number is the result of multiplying that number a specified number of times. Thus, 8 is the third power of 2 (2^3) and 81 is the fourth power of 3 (3^4).

It is quite possible to have a negative exponent, as in the case of 2^{-3}. A negative exponent indicates a reciprocal. The ***reciprocal*** of any number is one divided by that number. Thus,

the reciprocal of, say, 5 is $\frac{1}{5}$; and

2^{-3} represents $\frac{1}{2^{-3}} = \frac{1}{8} = 0.125$

Indices also do not have to be whole numbers – they can be ***fractional***, as in the case of $16^{1/4}$. This indicates that the base is raised to the power quarter. It is, though, more usual to refer to this as the fourth root of the base and to write it as follows:

$$\sqrt[4]{16}$$

The symbol √ indicates a root. A ***root*** is the number that produces a given number when raised to a specified power. Thus, the fourth root of 16 is 2 ($2 \times 2 \times 2 \times 2 = 16$).

A square root of any number is the number which, when multiplied by itself, is equal to the first number. Thus

$$\sqrt[2]{9} = 3 \quad (3 \times 3 = 9)$$

The ***root index*** (the little 4 or little 2 in the above examples) can be any number. It is, though, customary to omit the "little 2" from the root sign when referring to the square root. Thus

$\sqrt{36}$ refers to the square root of 36, which is 6.

Expressing Numbers in Standard Form

The exploration of indices leads us on to this very useful way of expressing numbers.

Working with very large – or very small – numbers can be difficult, and it is helpful if we can find a different way of expressing them which will make them easier to deal with. ***Standard form*** provides such a way.

Consider the number 93,825,000,000,000. It is a very big number and you would probably have great difficulty stating it, let alone multiplying it by 843,605,000,000,000.

In fact, can you even tell which is the bigger? Probably not. It is made a little easier by the two numbers being very close to each other on the page, but if one was on a different page, it would be very difficult indeed.

However, help is at hand.

The method of expressing numbers in standard form reduces the number to a value between 1 and 10, followed by the number of 10s you need to multiply it by in order to make it up to the correct size. The above two numbers, under this system are:

9.3825×10^{13} and 8.43605×10^{14}.

The "13" and "14" after the 10 are index numbers, as discussed above. Written out in full, the first number would be:

$9.3825 \times (10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10)$.

This is not particularly useful, so an easier way of dealing with a number expressed in standard form is to remember that the index number after the 10 tells you how many places the decimal point has to be moved to the ***right*** in order to create the full number (remembering also to put in the 0s as well). So, the second number from above, written out in full would be:

8	.	4	3	6	0	5	0	0	0	0	0	0	0	0	=	843,605,000,000,000
	1		*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	

Converting numbers in this way makes arithmetic operations much easier – something which we shall return to later in your studies – and also allows us to compare numbers at a glance. Thus, we can see immediately that 8.43605×10^{14} is larger than 9.3825×10^{13}.

Just as with indices, as above, we can also have numbers expressed in standard form using a negative index – for example, 2.43×10^{-2}.

Before we consider this in more detail, it would be helpful to recap exactly what the index number means.

We know that $10 \times 10 = 100$, so we can say that $10^2 = 100$.

We can go on multiplying 10 by itself to get:

$10^3 = 1,000$

$10^4 = 10,000$, etc.

We also know from above that, when there is a minus number in the index, it indicates a reciprocal – i.e. one divided by the number as many times as the index specifies. Thus:

$$10^{-1} = \frac{1}{10} = 0.1.$$

We could go on to say that $10^{-2} = 0.01$ and $10^{-3} = 0.001$, etc. Again, you will note that a minus index number after the 10 tells you how many places the decimal point has to be moved to the ***left*** in order to create the full number (remembering also to put in the 0s as well).

Now you should be able to work out that $2.43 \times 10^{-2} = 0.0243$.

Factorials

A factorial is the product of all the whole numbers from a given number down to one.

For example, 4 factorial is written as 4! (and is often read as "four bang") and is equal to $4 \times 3 \times 2 \times 1$.

You may encounter factorials in certain types of statistical operations.

Practice Questions 10

1. Express the following numbers in standard form

   (a) 234

   (b) 0.045

   (c) 6,420

   (d) 0.0005032

   (e) 845,300,000

   (f) 0.00000000000000003792

2. Write out in full the following numbers which are expressed in standard form.

   (a) 4.68×10^{2}

   (b) 9.35×10^{-2}

   (c) 1.853×10^{6}

   (d) 6.02×10^{-5}

   (e) 7.653×10^{12}

   (f) 8.16×10^{-10}

Now check your answers with the ones given at the end of the unit.

ANSWERS TO PRACTICE QUESTIONS

Practice Questions 1

1. (a) 4,000
   (b) 8,000
   (c) 7,000
2. (a) 7,000,000
   (b) 8,000,000
   (c) 5,000,000
3. (a) 2,840
   (b) 3,880
   (c) 7,890
   (d) 9,730
4. (a) Four
   (b) Three
   (c) Four
   (d) Two

Practice Questions 2

1. (a) 57,157
   (b) 232,140
   (c) 952
   (d) 554
2. The check on your answers is achieved by adding a column into which you put the sum of each row. The total of the check column should equal the sum of the four column totals.

	A	**B**	**C**	**D**	**Check**
	322	418	42	23	805
	219	16	600	46	881
	169	368	191	328	1,056
	210	191	100	169	670
Totals	920	993	933	566	3,412

3. (a) $(6 + 3) \times 6 + 4 \;=\; 9 \times 6 + 4 \;=\; 54 + 4 = 58$
   (b) $(6 + 3) \times (6 + 4) \;=\; 9 \times 10 \;=\; 90$
   (c) $10 \times (5 + 2) - (10 \div 5) \;=\; 10 \times 7 - 2 \;=\; 70 - 2 \;=\; 68$
   (d) $24 + 24 \div (2 \times 2) \;=\; 24 + 24 \div 4 \;=\; 24 + 6 \;=\; 30$
   (e) $6 \times (24 \div [4 \times 2]) \;=\; 6 \times (24 \div 8) \;=\; 6 \times 3 \;=\; 18$

(f) $(15 \div [3 + 2] + 6 + [4 \times 3] \times 2) \div 11$

$= (15 \div 5 + 6 + 12 \times 2) \div 11$

$= (3 + 6 + 24) \div 11$

$= 33 \div 11$

$= 3$

Practice Questions 3

1. 3
2. 3
3. 14
4. –2
5. –6
6. –6
7. –9
8. 15
9. 1
10. –100

Practice Questions 4

1. (a) $\frac{1}{4}$ (b) $\frac{1}{8}$

   (c) $\frac{3}{10}$ (d) $\frac{1}{3}$

   (e) $\frac{2}{7}$ (f) $\frac{1}{6}$

   (g) $\frac{4}{9}$ (h) $\frac{1}{3}$

   (i) $\frac{1}{8}$ (j) $\frac{3}{8}$

2. (a) $\frac{2}{10}$ (b) $\frac{12}{20}$

   (c) $\frac{36}{81}$ (d) $\frac{72}{88}$

3. (a) $2\frac{2}{5}$ (b) $2\frac{1}{8}$

   (c) $4\frac{1}{6}$ (d) $10\frac{1}{3}$

4. (a) $\frac{8}{3}$ (b) $\frac{27}{8}$

(c) $\frac{33}{8}$ (d) $\frac{39}{4}$

Practice Questions 5

1. $\frac{7}{12}$
2. $\frac{4}{15}$
3. $\frac{19}{24}$
4. $\frac{11}{30}$
5. $1\frac{3}{20}$
6. $\frac{17}{24}$
7. $\frac{5}{8}$
8. $\frac{5}{12}$
9. $\frac{1}{2}$
10. $\frac{27}{40}$
11. $\frac{8}{21}$
12. $\frac{3}{8}$
13. $4\frac{1}{4}$
14. $8\frac{5}{8}$
15. $7\frac{11}{30}$
16. $1\frac{1}{12}$
17. $2\frac{2}{9}$
18. $3\frac{19}{24}$
19. $2\frac{7}{24}$
20. $3\frac{11}{12}$

Practice Questions 6

1. $\frac{21}{4} = 5\frac{1}{4}$
2. $\frac{16}{3} = 5\frac{1}{3}$
3. $\frac{15}{6} = 2\frac{1}{2}$
4. $\frac{18}{7} = 2\frac{4}{7}$
5. $\frac{2}{18} = \frac{1}{9}$
6. $\frac{3}{24} = \frac{1}{8}$
7. $\frac{7}{24}$
8. $\frac{5}{28}$

Practice Questions 7

1. (a) 11.68 (b) 19.47

(c) 26.413 (d) 23.092

(e) 5.42 (f) 5.178

(g) 6.591 (h) 9.17

	(i)	10.136		(j)	15.84
	(k)	23.78		(l)	3.276
	(m)	70		(n)	1.6
	(o)	5.6		(p)	50
2.	(a)	0.40		(b)	0.375
	(c)	0.75		(d)	0.875
	(e)	0.625		(f)	0.833
	(g)	0.571		(h)	0.313
3.	(a)	$\frac{125}{1000} = \frac{1}{8}$		(b)	$\frac{60}{100} = \frac{3}{5}$
	(c)	$\frac{175}{100} = 1\frac{3}{4}$		(d)	$\frac{750}{1000} = \frac{3}{4}$

Practice Questions 8

1.	(a)	3%		(b)	37.5%
2.	(a)	$\frac{1}{50}$ 2%		(b)	$\frac{2}{25}$
	(c)	$\frac{3}{40}$		(d)	$\frac{9}{50}$
3.	(a)	£30		(b)	90°
	(c)	35 bottles		(d)	£483.30
	(e)	£24.15			
4.	(a)	20%		(b)	33.33%
	(c)	2.5%		(d)	150%

Practice Questions 9

1. Firm A: 1 : 4

   Firm B: 1 : 9

   Firm C: 1 : 3

   Firm D: 1 : 5

2. (a) X: £15,000
   Y: £20,000
   Z: £25,000

   (b) F: £1,800
   G: £2,700
   H: £2,700

Practice Questions 10

1. (a) 2.34×10^{2}
   (b) 4.5×10^{-2}
   (c) 6.42×10^{3}
   (d) 5.032×10^{-4}
   (e) 8.453×10^{8}
   (f) 3.792×10^{-15}

2. (a) 468
   (b) 0.0935
   (c) 1,853,000
   (d) 0.0000602
   (e) 7,653,000,000,000
   (f) 0.000000000816

Study Unit 2

Algebra, Equations and Formulae

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INTRODUCTION

In this study unit we continue our review of basic concepts by looking at algebra. Algebra essentially involves the substitution of letters for numbers in calculations, so that we can establish rules and procedures for carrying out mathematical operations which can be applied whatever the actual numbers involved. In this sense, then, algebra enables general expressions to be developed and general results to be obtained.

We shall then go on to introduce the subject of equations. An equation is simply a statement of equality between two mathematical expressions, and is generally used to enable one or more unknown quantities to be worked out. There are various rules and procedures which can be applied to equations to help this process and these will be examined in full. Here we shall be concerned with "simple" equations, but later in the course we shall consider more complex forms. However, the same basic rules are applied, so it is important to understand these at the outset.

Finally, we look briefly at formulae. A formula is one particular form of generalised mathematical statement, usually expressed in the form of an equation, which sets out a rule which may be applied in particular situations. It enables us to work out an unknown quantity or value provided we know certain specific quantities or values. There are formulae which describe a great many physical situations or mathematical problems and you will encounter these throughout your studies, both in this course and many others.

Objectives

When you have completed this study unit you will be able to:

- outline the basic principles of algebra;
- apply the basic arithmetic operations of addition, subtraction, multiplication and division to algebraic notation, and simplify algebraic expressions by the process of collecting like terms;
- define equations and formulae and outline their uses;
- find unknown quantities and values from simple equations by using transposition;
- outline the principles of formulae and how they are constructed;
- rearrange the terms in a formula to isolate different unknowns.

A. SOME INTRODUCTORY DEFINITIONS

Algebra

Algebra is a branch of mathematics in which, instead of using numbers, we use letters to represent numbers.

We all know that 2 + 3 = 5.

Suppose, though, that we substitute letters for the first two numbers, so that:

$2 = a$

$3 = b$

We can then write:

$a + b = 5$

All that has happened is that we have replaced the numbers with letters. However, a number is a specific quantity – for example, 5 is more than 4, but less than 6 – whereas a letter can be used to represent any number. Thus, in the above expression, "a" could be 4 and "b" could be 1. We only know that they are 2 and 3 respectively because we defined them as such before.

The main consequence of this is that algebra uses general expressions and gives general results, whereas arithmetic (using numbers) uses definite numbers and gives definite results. Arithmetic is specific, whereas algebra is general.

Let us consider some examples to illustrate this further.

- Suppose you have a piece of wood which is 7 metres long and from it you wish to cut a piece 4 metres long. The length of the remaining piece is 3 m, calculated as follows:

  $7 - 4 = 3$

  This is a specific arithmetic statement relating to cutting a specific amount from this particular piece of wood.

  We could translate this into an algebraic expression by substituting letters for the specific lengths:

  let the original length of the piece of wood = x metres

  the length of the piece cut off = y metres

  The calculation can now be shown as:

  $x - y = 3$

  This is now a general statement for cutting one length of wood from another to leave a piece 3 metres in length.

- To find the area of a floor measuring 10 m long and 9 m wide, we multiply one dimension by the other:

  Area $= 10 \times 9$ square metres

  $= 90$ sq m

  Substituting the letters "l" and "w" to represent the actual length and width, we can reformulate the expression as:

  Area $= l \times w$

Again, this is a general expression which can be made specific by putting in particular values for "l" and "w".

- The distance travelled by a train in 3 hours at a speed of 60 miles per hour is easily calculated as:

  $$3 \times 30 = 180 \text{ miles}$$

  Here again, letters may be substituted to give us a general expression:

  $$s \times t = d$$

  where: speed = s mph

  time = t hours

  distance travelled = d miles

Thus, using algebra – working with letters instead of numbers – allows us to construct general mathematical expressions. This is not particularly helpful in itself, but is very important when we come to consider equations.

Equations

An equation is simply a mathematical statement that one expression is equal to another. So, for example, the statement that "2 + 2 = 4" is an equation.

Note that an equation has ***two sides***. Here, they are "2 + 2" and "4". The two sides must always be in equality for the statement to be an equation.

If we now introduce the concept of algebra into this, we have the makings of an extremely useful mathematical tool.

For example, given that a certain number multiplied by 3 is 6, we can write this as:

$$3 \times \text{the required number} = 6.$$

To save writing "the required number" (or the " unknown" number), it is more convenient to call it "x", so that we can then write:

$$3 \times x = 6$$

Here are a few simple statements which you can easily write as equations for practice:

(a) a certain number is added to 4 and the result is 20.

(b) a certain number is multiplied by 4 and the result is 20.

(c) if 4 is taken from a certain number the result is 5.

(d) if a certain number is divided by 3 the result is 1.

The correct equations are:

(a) $x + 4 = 20$

(b) $x \times 4 = 20$

(c) $x - 4 = 5$

(d) $x \div 3 = 1$

The point of constructing equations in this way is that provide a means by which you can work out the unknown value (here, "x"). These examples are very simple and you can probably solve them (i.e. work out the unknown) very easily. However, equations can get very complicated. Don't worry, though – there are a number of simple rules which can be used to help solve them.

Formulae

A mathematical formula (plural "formulae") is a special type of equation which can be used for solving a particular problem.

In fact, we have already introduced two formulae in the preceding section on algebra:

- Area = length $\times$ width
- Distance travelled = speed $\times$ time

The essence of these two statements is that they are always true, whatever the actual values. Thus, a formula is an equation which always applies to a particular mathematical problem, whatever the actual values. It provides a set of rules which can be used in a particular situation in order to solve the problem

If we know the actual values, we could put them into the equation, but if not we would have to treat them as unknown. Thus, we can formulate a general statement using algebra as follows:

- $A = l \times w$

  where: A = area

  l = length

  w = width

- $d = s \times t$

  where: speed = s mph

  time = t hours

  distance travelled = d miles

Having a formula for a particular problem set out as a general equation using algebra enables us to work out the answer by substituting actual values back into it for the unknowns.

For example, if we know we have been driving for 2 hours at an average speed of 60 mph, we can easily work out the distance travelled by substituting 60 for "s" and 2 for "t" in the above formula:

$$\begin{aligned} d &= s \times t \\ &= 60 \times 2 \\ &= 120 \text{ miles} \end{aligned}$$

There are many formulae used in mathematics and other physical sciences such as chemistry or physics. You will meet a good many in your studies – principally in this subject and in financial subjects. Again, do not worry too much about them. They are usually quite straightforward and, by using the general rules governing equations, can easily be used to solve the problem faced.

B. ALGEBRAIC NOTATION

As algebraic letters simply represent numbers, the operations of addition, subtraction, multiplication and division are still applicable in the same way. However, in algebra it is not always necessary to write the multiplication sign. So, instead of "$a \times b$", we would write simply "ab" (or sometimes "a.b", using the full stop to represent multiplication).

Addition and Subtraction

Addition and subtraction follow the same rules as in arithmetic. Thus, just as:

$$3 + 3 + 3 + 3 + 3 + 3 = 3 \times 6$$

so $\quad y + y + y + y + y + y = y \times 6$, or $6y$

You should note that this holds good only for ***like terms***. Like terms are those of the same symbol or value – for example, all the terms to be added above were "y".

When ***unlike*** terms are to be added, we can simplify an arithmetical statement, but not an algebraic statement. For example:

$$3 + 6 + 7 = 16$$

$$y + x + z = y + x + z$$

The three numbers, when added, can be reduced to a simpler form – i.e. 16. However, there is not simpler means of expressing "x + y + 3".

We can, though, simplify expressions when they involve like terms.

Just as the result of adding two numbers does not depend upon their order, so the result of addition in algebra is not affected by order. Thus:

$$2 + 7 = 7 + 2 = 9$$

and $\quad x + y + 3 = 3 + y + x = y + 3 + x$, etc.

In simplifying an algebraic expression, we can collect like terms together whenever possible. For example:

$$5x + 2y - x + 3y + 2 = (5x - x) + (2y + 3y) + 2 = 4x + 5y + 2$$

We have collected together all the "x"s $(5x - x)$ and all the "y"s $(2y + 3y)$, but cannot do anything with the "2" which is unlike the other terms.

This distinction between like and unlike terms, and the way in which they can be treated is an important point.

Multiplication

As we have seen above, the product of a number and a letter, for example $3 \times a$, may be written as 3a. Always place the number before the letter, thus 3a, *not* a3. Since multiplication by 1 does not alter the multiplicand, the expression 1a is not used; you just write a.

To multiply unlike terms, we simply write them together with no sign. Thus:

$$a \times b \times c \times d = abcd$$

If numerical factors are involved then they are multiplied together and placed in front of the simplified term for the letters:

$$3a \times 2b \times 4c \times d = 24abcd$$

That this is correct can be shown by writing it in full and changing the order of multiplication:

$$3 \times a \times 2 \times b \times 4 \times c \times d$$

$$= 3 \times 2 \times 4 \times a \times b \times c \times d$$

$$= 24abcd$$

Multiplying like terms (for example, $x \times x$) gives a result of the term being raised to the power of the multiplier (here, x^2). We shall consider this in more detail below.

Division

Division should present little difficulty if we follow the rules used in arithmetic. Consequently, just as we write:

$\frac{4}{2}$ for $4 \div 2$

so we can write

$\frac{x}{y}$ for $x \div y$

Remember that any number goes into itself once. This will prevent you from saying that $6 \div 6 = 0$ instead of $6 \div 6 = 1$.

In the same way $x \div x = 1$. It is then easy to cancel algebraic fractions. For example:

$\frac{2xy}{4x}$ can be cancelled by dividing both the numerator and denominator by $\frac{x}{x}$ to give $\frac{2y}{4}$

$\frac{2y}{4}$ can be further cancelled by dividing both the numerator and denominator by 2 to give $\frac{y}{2}$

or we could simply say we can cancel $\frac{2xy}{4x}$ by dividing both parts by 2x.

We can always cancel like terms in the numerator and denominator.

Indices, Powers and Roots

You will remember that indices are found when we multiply the same number by itself several times. The same rules apply in algebra, but we can investigate certain aspects further by using algebraic notation.

To recap,

$x \times x$	is written as x^2	(referred to as "x squared")
$x \times x \times x$	is written as x^3	(referred to as "x cubed")
$x \times x \times x \times x$	is written as x^4	(referred to as "x to the power four" or "x raised to the fourth power")

and so on.

In, for example, x^5, x is termed the base and 5 is called the exponent or index of the power.

Note two particular points before we go on:

$x = x^1$

$x^0 = 1$

(a) Multiplication of indices

You will note that, for example:

$$\begin{aligned} x^2 \times x^4 &= (x \times x) \times (x \times x \times x \times x) \\ &= x \times x \times x \times x \times x \times x \\ &= x^6 \\ &= x^{2+4} \end{aligned}$$

and $x^3 \times x^5 = (x \times x \times x) \times (x \times x \times x \times x \times x)$

$= x \times x \times x \times x \times x \times x \times x \times x$

$= x^8$

$= x^{3+5}$

There is in fact a general rule that, when we have two expressions with the same base, multiplication is achieved by **adding** the indices. Thus:

$$x^m \times x^n = x^{m+n} \qquad \textit{(Rule 1)}$$

(b) Division of indices

When we divide expressions with the same base, we find we can achieve this by **subtracting** the indices. For example:

$$\frac{x^6}{x^3} = \frac{x \times x \times x \times x \times x \times x}{x \times x \times x}$$

$= x \times x \times x$

$= x^3$

$= x^{6-3}$

and $\dfrac{x^2}{x} = \dfrac{x \times x}{x}$

$= x$

$= x^1$

$= x^{2-1}$

The general rule is:

$$\frac{x^m}{x^n} = x^{m-n} \qquad \textit{(Rule 2)}$$

(c) Showing reciprocals

In Rule 2 above, let us put m = 0, so that the expression becomes :

$$\frac{x^0}{x^n} = x^{0-n}$$

We know that $x^0 = 1$, so

$$\frac{1}{x^n} = x^{0-n} = x^{-n}$$

Thus, negatives denote reciprocals.

(d) Raising one power by another

To work out an expression such as $(x^2)^3$ – i.e. x squared all cubed – we have $x^2 \times x^2 \times x^2$ as it is the cube of x^2.

Thus $(x^2)^3 = x^2 \times x^2 \times x^2$

$= x \times x \times x \times x \times x \times x$

$= x^6 = x^{2 \times 3}$

In general, we have $(x^m)^n = x^{mn}$ *(Rule 3)*

(e) Roots

Indices do not have to be whole numbers. They can be fractional.

Let us consider $x^{½}$ (i.e. x raised to the power half). We know from rule (3) above that:

$$(x^{½})^2 = x^{½ \times 2} = x$$

Thus, $x^{½}$ when multiplied by itself gives x. However, this is exactly the property that defines the square root of x. Therefore, $x^{½}$ is the **square root of x** which you will sometimes see denoted as $\sqrt[2]{x}$ or, more commonly, as $\sqrt{x}$.

Therefore, $x^{½} \equiv \sqrt{x}$. (The symbol "≡" means identical with.)

Similarly, $(x^{1/3})3 = x$. Thus, $x^{1/3}$ is the **cube root** of x which is also denoted by the symbol $\sqrt[3]{x}$.

Similarly, $x^{¼}$ is the **fourth root** of x denoted by $\sqrt[4]{x}$.

Thus, in general, we can say that:

$x^{1/n}$ is the nth root of x or $\sqrt[n]{x}$.

Note that in all the above expressions the symbol to the left of the root sign – for example, $\sqrt[3]{x}$ – is written so that it is level with the top of the root sign. A number written level with the foot of the root sign – for example, $3\sqrt{x}$ – would imply a multiple of the square root of x – in this example, 3 times the square root of x.

We shall often need to use square roots in later work, particularly in statistics and in stock control.

(f) Roots of powers

There is one more type of index you may meet – an expression such as:

or $x^{5/3}$, or in general $x^{m/n}$.

To interpret these, remember that the numerator of a fractional index denotes a power and that the denominator denotes a root.

Thus, $x^{3/2}$ is $(x^3)^{½}$, i.e. the square root of (x cubed) or $\sqrt[2]{x^3}$. Alternatively, it is $(x^{½})^3$, i.e. the cube of (the square root of x) or $(\sqrt{x})^3$.

Let us consider its value when x = 4:

$$4^{3/2} = \sqrt[2]{4^3} = \sqrt{4 \times 4 \times 4} = \sqrt{64} = 8 \ \text{(since } 8 \times 8 = 64\text{)}$$

or $$4^{3/2} = (\sqrt{4})^3 = 2^3 = 2 \times 2 \times 2 = 8$$

Therefore, it does not matter in which order you perform the two operations, and we have:

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

(g) Collecting like terms

We saw earlier how we can collect **like** items. Let us look at a couple of examples where indices are involved. It is no more complicated, even though it may look that way.

$$a^2 + 2a^2 = a^2 + a^2 + a^2 = 3a^2$$

These are like terms and can be added.

$b^2 + 2b^2 + 2b^3 = b^2 + b^2 + b^2 + b^3 + b^3$

$= 3b^2 + 2b^3$

Note that b^2 and b^3 are ***not*** like terms, so they cannot be combined.

Brackets in Algebra

The rules for brackets are exactly the same in algebra as in arithmetic. However, if we have unlike terms inside the brackets, it is not possible to collect them together before removing the brackets. In such cases, we need to be very careful when removing them and use the following three rules

- if the bracket has a term (a number or a letter) in front of it, everything within the bracket must be multiplied by that term;
- if a bracket has a + sign before it, or before the term by which the terms inside the brackets are to be multiplied, the signs + or – within the brackets remain unchanged;
- if a bracket has a – sign before it, or before the term by which the terms inside the brackets are to be multiplied, the signs + or – within the brackets are changed so that + becomes –, and vice versa;

These rules need a little explanation, so let us consider some examples.

(a) $4x + 2(x - y)$

The terms inside the brackets are unlike and cannot be further collected.

All the terms within the brackets must be multiplied by the term outside before removing the brackets, so this gives:

$4x + 2x - 2y = 6x - 2y$

(b) $x - (y + 3z)$

Again, the terms inside the brackets are unlike and cannot be further collected, so we can proceed to remove the brackets as the next step.

Since there is a – sign before the brackets, we must reverse the signs of the terms inside when removing them. Thus:

$x - y - 3z$

We can clarify this by imagining that there is an unwritten "1" before the bracket, making the expression, effectively:

$x - 1(y + 3z)$

This "1" is multiplying the whole bracket:

$x - 1 \times (y \times 3z)$

This gives us three separate terms:

x, $-1 \times y$, and $-1 \times 3z$

Thus, we have:

$x - y - 3z$

(c) $3a(a - 3) - 4b(b - 4)$

The first bracket is multiplied by 3a and the second by –4b. This gives us:

$3a^2 - 9a - 4b^2 + 16b$

Note that the – sign before the second bracket changes the + sign inside.

Often we find it necessary to insert brackets into an algebraic expression. This is known as ***factorisation*** because, when we include the brackets, we place any common factor outside the brackets.

Thus:

$$x + 5y + 5z = x + 5(y + z)$$

$$x - 5y + 5z = x - 5(y - z)$$

Notice how we must keep an eye on the signs as well.

Practice Questions 1

Simplify the following expressions, writing "no simpler form" if you think any cannot be further simplified:

1. $a + a + a + a$
2. $ab + ab$
3. $3x + 2x + x$
4. $3x + 2y + y$
5. $5x - 3x + x$
6. $7xy - 2x + y$
7. $3a - 2a$
8. $a + a + b + a - 3$
9. $x + y + xy$
10. $x + x + x + y + y + y$
11. $a + 2b + 3a + 4b + 5$

12 $3x - (x - 3)$

13 $x + 2(y + z)$

14. $4 - 3(-2x + y) + (-x - y)$
15. $x - (y - 3z)$
16. $4x + 2(x - y)$
17. $a(b - c) - b(c - a)$
18. $3a(a - 3) - 4b(b - 4)$

Now check your answers with the ones given at the end of the unit.

C. SOLVING EQUATIONS

When we talk about solving an equation, we mean finding the value of the unknown or unknowns using the other numbers in the equation. For example, in:

$$4x = 20$$

the value of the unknown (x) can be discovered because of the stated relationship between 4 and 20. You will easily conclude that if 4 times the unknown = 20 then the unknown **must** be 5.

There are a number of rules which can be used to help solve equations and we shall consider these below in relation to equations which have only one unknown. In a later unit we shall examine more complex equations – those with two or more unknowns – and consider additional rules. However, these are based on the simple rules which we set out here.

Equality of Treatment to Both Sides of the Equation

This is the golden rule that you must always follow when handling equations and there are no exceptions.

What you do to one side of the equation, you must also do to the other.

Remember that an equation has two sides and those two sides must always be in equality for the whole statement to be an equation. So, anything we do to the equation must maintain this equality.

Another way of thinking about this is that an equation is like a balance. If the weights of a balance are equal on both sides it is “in balance”. You can add an equal weight to each side, or take an equal weight from each side, and it will remain “in balance”.

Looking at the rule in more detail, we can see a number of possible operations which may be carried out without changing the equality of the equation.

(a) The same number may be added to both sides of the equation

For example:

$$x - 4 = 6$$

If we add 4 to each side, then we get:

$$x - 4 + 4 = 6 + 4$$

$$x = 10$$

(b) The same number may be subtracted from both sides of the equation

For example:

$$x + 6 = 18$$

If we deduct 6 from each side, we get:

$$x + 6 - 6 = 18 - 6$$

$$x = 12$$

(c) Both sides of the equation may be multiplied by the same number.

For example:

$$\tfrac{1}{3} \text{ of a number} = 10, \text{ or } \frac{1}{3}x = 10, \text{ or } \frac{x}{3} = 10$$

If we multiply both sides by 3, we get:

$$\frac{x}{3} \times 3 = 10 \times 3$$

$$\frac{3x}{3} = 30$$

$$x = 30$$

(d) Both sides of the equation may be divided by the same number.

For example:

2 times a number is 30, or $2x = 30$

If we divide both sides by 2, we get:

$$\frac{2x}{2} = \frac{30}{2}$$

$$x = 15$$

You should have noticed that the object of such operations is to isolate a single unknown, x, on one side of the equals sign.

Practice Questions 2

Simplify the following expressions, writing "no simpler form" if you think any cannot be further simplified:

1. $x - 7 = 21$
2. $x - 3 = 26$
3. $x + 9 = 21$
4. $\frac{x}{5} = 11$
5. $\frac{x}{8} = 7$
6. $3x = 30$

Now check your answers with the ones given at the end of the unit.

Transposition

When you understand the rules in the previous subsection you may shorten your working by using transposition instead.

Transposition is a process of transferring a quantity from one side of an equation to another by changing its sign of operation. **This is done so as to isolate an unknown quantity on one side.**

You must observe the following rules:

- **An added or subtracted term may be transposed from one side of an equation to the other if its sign is changed from + to −, or from − to +**

  Consider the following examples:

  (a) $x - 4 = 6$

  Transfer the 4 to the right-hand side of the equation and reverse its sign:

  $x = 6 + 4$

  $x = 10$

  (b) $x + 6 = 18$

  Transfer the 6 to the right-hand side and reverse its sign:

  $x = 18 - 6$

  $x = 12$

  (c) $12 + x = 25$

  Transpose the 12 and reverse its sign:

  $x = 25 - 12$

  $x = 13$

  (d) $18 - x = 14$

  First transpose the x and reverse its sign:

  $18 = 14 + x$

  Then transpose the 14 and reverse its sign:

  $18 - 14 = x$

  $x = 4$

  Note that we can keep transposing terms from one side to the other until we have isolated the unknown quantity.

- **A multiplier may be transposed from one side of an equation by changing it to the divisor on the other**

  Similarly, a divisor may be transposed from one side of an equation by changing it to the multiplier on the other

  Consider the following examples:

  (a) $\frac{x}{4} = 12$

  To get x by itself, move 4 across the = sign and change it from divisor to multiplier, i.e.:

  $x = 12 \times 4$

  $x = 48$

  (b) $2x = 20$

  To get x by itself, transpose the 2 and change it from multiplier to divisor, i.e.:

  $x = \frac{20}{2}$

  $x = 10$

(c) $\frac{24}{x} = 6$

In this case, we need to transpose x and change it from divisor to multiplier, i.e.:

$$24 = 6x$$

Then, to get x by itself, we transpose the multiplier (6) and change it to a divisor:

$$\frac{24}{6} = x$$

$$x = 4$$

This rule is called **cross multiplication**.

Practice Questions 3

Find the value of x by transposition:

1. $\frac{x}{6} = 36$
2. $\frac{72}{x} = 24$
3. $3x + 3 = 21$
4. $5x = 25$
5. $2x - 3 = 15$
6. $21 - x = 15$

Now check your answers with the ones given at the end of the unit.

Equations with the Unknown Quantity on Both Sides

These equations are treated in the same way as the other equations we have met so far. We simply keep transposing terms as necessary until we have collected all the unknown terms on one side of the equation.

For example, consider $3x - 6 = x + 8$

First move the –6 to the right-hand side, making it +6:

$$3x = x + 8 + 6$$

Then transpose x from right to left, changing its sign:

$$3x - x = 8 + 6$$

$$2x = 14$$

Finally, move the 2 to the right-hand side changing it from multiplier to divisor, and thus isolating x on the left:

$$x = \frac{14}{2}$$

$$x = 7$$

Practice Questions 4

Find the value of x:

1. $x + 3 = 8 - 2x$
2. $2x = 24 - x$
3. $\frac{1}{2}x = 15 - x$
4. $5x - 3 = 27 + 2x$
5. $18 + x = 5x - 2$
6. $\frac{2x}{3} = 24 + 6x$

Now check your answers with the ones given at the end of the unit.

D. FORMULAE

A formula is a mathematical model of a real situation.

The easiest way to explain this is to give an example.

Formulating a Problem

When person borrows money, he/she has to pay interest on the loan (called the "principal"). The amount of interest payable is determined by the annual rate of interest charged on the money borrowed, taking into account the time over which the loan is repaid. If we want to work out the amount of interest repayable on a particular loan, the calculation is as follows:

Interest = the principal multiplied by the percentage rate of interest, multiplied by the number of time periods the interest is to apply.

We could model this situation by using algebraic notation and then show the whole calculation as an equation:

let: I = simple interest

P = principal

r = rate of interest (%)

n = number of periods

then: $I = P \times \frac{r}{100} \times n$

This equation is the formula for calculating simple interest. It can be used for any simple interest calculation and enables us to find the value of any one of the elements, provided that we know all the others.

Thus, what we have done is express the real situation as a mathematical model using an algebraic equation.

This particular formula is very important in financial studies and can be applied in many situations. However, it is possible to develop formulae to model any situation involving mathematical relationships. Some are just useful for helping to understand how to work out a particular problem or situation you are faced with, whilst others describe key mathematical relationships which you will meet time and again throughout the course.

For example, consider the following situation:

A man runs at a certain number of km per hour for a certain number of hours, and then cycles at a certain number of km per hour for a certain number of hours. What is his average speed?

let: x = the speed at which he runs (in kph)

y = the number of hour he runs

p = the speed at which he cycles (in kph)

q = the number of hour he cycles

Therefore, we can say he covers (xy + pq) kilometres in (y + q) hours.

then: Average speed $= \dfrac{xy + pq}{y + q}$ km per hour.

Note that when we use a letter such as x to represent an unknown quantity, it only represents that quantity and the units must be exactly specified. For example, we said "let x = the speed at which he cycles (in kph)" – to have said "let x = the speed at which he cycles" would have been inaccurate.

The steps to take in formulating the problem, then, are as follows:

- select a letter to represent each of the quantities required;
- convert each block of information from the problem into an algebraic term or statement, using the letters selected;
- form an equation by combining the symbolic terms, ensuring that all terms are expressed in the same units.

The last step is to test the formula by going back to the problem and check that it works by using real number values in the formula through the process of substitution. We shall look at substitution below.

As practice, see if you can work the following problem out by developing a formula. Note that the unknown quantities have already been given letters.

A grocer mixes m kg of tea at x pence per kg with n kg of tea at y pence per kg. If he sells the mixture at v pence per kg, what profit, in pence, does he make? Bring your answer to £s.

The solution would be as follows:

Profit = Selling price – Cost price

Selling price = Quantity sold $\times$ Price

v(m + n) pence

Cost price = mx + ny pence

Profit = v(m + n) – (mx + ny) pence

To convert this to £s, we simply divide the result by 100:

$$\text{Profit} = £\,\frac{v(m + n) - (mx + ny)}{100}$$

Constants and Variables

As we have seen above, formulae consist of letters (algebraic notation) and, sometimes, numbers. These are combined together in particular ways to express the mathematical relationships between the terms such that a particular result is obtained.

The numbers in a formula always stay the same and are known as ***constants***. so, for example, in the formula for simple interest, the constant will always be 100, no matter what the values given to the P, r and n in a particular situation.

The letters are, on the other hand variables. The value that they have in working out the formula in any particular situation may vary – it depends on the particular circumstances of the problem.

Substitution

We defined a formula as a mathematical model of a real situation. In order to use that model in a real situation, we have to give the variables the values which apply in that particular situation. Doing this is the process of substitution.

Turning back to the formula for simple interest, consider the following example:

What is the simple interest on £1,000 invested at 10% per annum (i.e. per year) for two years?

First of all we should restate the formula:

$$I = P \times \frac{r}{100} \times n$$

Now we substitute the values from the problem into the formula:

$$P = £1{,}000$$

$$r = 10$$

$$n = 2$$

So: $$I = £1{,}000 \times \frac{10}{100} \times 2$$

$$\frac{£1{,}000 \times 10 \times 20}{100}$$

$$£200$$

This explains the principle of substitution. In the next unit you will get much more practice in this.

Rearranging Formulae

Finally, we should note that a formula can be rearranged to find a different unknown quantity. So, for example, when considering simple interest, we may wish to know how long it would take to earn a specified amount of interest from a loan made to someone else, at a particular rate of interest.

The way in which the formula is expressed at present does not allow us to do that, but by using the rules we discussed above in relation to simple equations, we can rearrange the formula so that it does.

Again, we start by restating the original formula:

$$I = P \times \frac{r}{100} \times n$$

Note that we can also express this as:

$$I = \frac{Prn}{100}$$

To find n, we need to rearrange the formula so that n is isolated on one side of the equation.

We can do this by dividing both sides by $P \times \frac{r}{100}$ or, as it may also be expressed, $\frac{Pr}{100}$:

$$\frac{I}{\left(\frac{Pr}{100}\right)} = \frac{\left(\frac{Pr}{100}\right)n}{\left(\frac{Pr}{100}\right)}$$

$$n = \frac{I}{\left(\frac{Pr}{100}\right)}$$

$$n = I \div \frac{Pr}{100} = I \times \frac{100}{Pr}$$

$$n = \frac{100I}{Pr}$$

Don't worry too much about the mechanics of this for now – again you will have plenty of practice at these types of operation as you work through the course. However, see if you can rearrange the simple interest formula to isolate the other two unknowns.

Practice Questions 5

Rearrange the simple interest formula to allow the following calculations to be made:

1. To find P – the principal which would provide a certain amount of interest at a particular rate of interest over a specified period.
2. To find r – the rate of interest which would have to be charged to earn a particular amount of interest on a specified principal over a certain period of time.

Now check your answers with the ones given at the end of the unit.

ANSWERS TO PRACTICE QUESTIONS

Practice Questions 1

1. $4a$
2. $2ab$
3. $6x$
4. $3x + 3y$
5. $3x$
6. No simpler form
7. a
8. $3a + b - 3$
9. No simpler form
10. $3x + 3y$
11. $4a + 4b + 5$

12 $3x - x + 3 = 2x + 3$

13 $x + 2y + 2z$

14. $4 + 6x - 3y - x - y = 4 + 5x - 4y$
15. $x - y - 3z$
16. $4x + 2x - 2y = 6x - 2y$
17. $ab - ac - bc + ab = 2ab - ac - bc$
18. $3a^2 - 9a - 4b^2 + 16b$

Practical Questions 2

1. $x = 28$
2. $x = 29$
3. $x = 12$
4. $x = 55$
5. $x = 56$
6. $x = 10$

Practice Questions 3

1. $x = 36 \times 6 = 218$
2. $72 = 24x, \quad \frac{72}{24} = x, \quad x = 3$
3. $3x = 21 - 3, \quad x = \frac{18}{3}, \quad x = 6$

4. $x = \frac{25}{5} = 5$

5. $2x = 15 + 3, \quad x = \frac{18}{2} = 9$

6. $21 = 15 + x, \quad 21 - 15 = x, \quad x = 6$

Practice Questions 4

1. $x + 2x = 8 - 3, \quad x = \frac{5}{3} = 1\frac{2}{3}$

2. $2x + x = 24, \quad x = 8$

3. $\frac{1}{2}x + x = 15, \quad x = 10$

4. $5x - 2x = 27 + 3, \quad x = 10$

5. $18 + 2 = 5x - x, \quad x = 5$

6. $-24 = 6x - \frac{2x}{3} = 5\frac{1}{3}x, \quad x = -24 \div 5\frac{1}{3} = -24 \times \frac{3}{16} = \frac{72}{16} = \frac{9}{2} = -4½$

Practice Questions 5

1. We always start by restating the original formula:

$$I = P \times \frac{r}{100} \times n \quad \text{or} \quad I = \frac{Prn}{100}$$

To isolate P, we divide both sides by $\frac{r}{100} \times n$ or $\frac{rn}{100}$

$$\frac{I}{\left(\frac{rn}{100}\right)} = \frac{P\left(\frac{rn}{100}\right)}{\left(\frac{rn}{100}\right)}$$

$$\frac{I}{\left(\frac{rn}{100}\right)} = P$$

$$P = I \div \frac{rn}{100} = I \times \frac{100}{rn}$$

So $\quad P = \frac{100I}{rn}$

2. Once again, start with the original formula

$$I = P \times \frac{r}{100} \times n \quad \text{or} \quad I = \frac{Prn}{100}$$

To find $\frac{r}{100}$, we divide both sides by $P \times n$:

$$\frac{I}{Pn} = \frac{\left(\frac{r}{100}\right)Pn}{Pn}$$

$$\frac{I}{Pn} = \frac{r}{100}$$

$$\frac{r}{100} = r\%. \text{ which is what we want to find}$$

so $$r\% = \frac{I}{Pn}$$

Study Unit 3

Basic Business Applications

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INTRODUCTION

In this unit we shall apply some of the numerical concepts introduced in the previous two units to some basic problems encountered regularly in the business world. The intention here is not to go into detail about the nature of these problems, but to concentrate on the principles of calculation involved in solving them. Thus, in most cases, we shall use very simplified situations and limited amounts of information.

At first sight, some of these business applications may seem quite hard to work out, but if you follow the logical steps shown here to breaking down the problem, applying the correct formulae and setting out the calculations clearly, you should not find them difficult in practice.

Objectives

When you have completed this study unit you will be able to:

- explain the rate of exchange between different currencies and convert values in one currency into another;
- explain discounts and commissions and calculate them from given information;
- state the formulae for simple interest and compound interest, and use them to calculate any of the amount of interest payable (or receivable), the principal, the rate of interest or the time period covered, given information about the other variables;
- explain depreciation and calculate it according to the straight-line method and the reducing-balance method;
- outline the way in which gross pay is made up and the main deductions made from gross pay in order to leave net or take-home pay;
- calculate the income tax payable on pay, taking into account non-taxable items and personal tax allowances.

A. CURRENCY AND RATES OF EXCHANGE

Each country has its own system of coinage and this is known as its **currency.** The process of finding the value of the currency of one country in terms of the currency of another country is called **exchange.**

Rates of Exchange

You are, no doubt, familiar with various currency systems. They are invariably based on the decimal number system – for example, America uses the dollar ($) and the cent, which is equivalent to one-hundredth of a dollar, France has the French franc (FF) and the centime, which is one-hundredth of a franc, and British currency comprises the pound sterling (£) and pence, which are one-hundredth of a pound.

The values of the different currencies used to depend upon the amount of gold which the country possessed, but now that we are no longer on the "gold standard", their values depends on the political and economic circumstances of the countries concerned. Such value is measured by how much one unit of currency costs in another currency. For example, one unit of British currency (£1) costs approximately 1.45 units of US currency ($1.44). This relationship of one currency to another is known as the ***rate of exchange***.

These values vary from day to day and are quoted in most daily papers. The changes in value are brought about by such things as changes in government, changes in government policies, the national budget, the relationship between the value of exports and imports, etc. To give you some idea of the scale of changes over time, before the Second World War the value of the dollar was about 21p (i.e. $1 = £0.21), whereas now it is about 69p (i.e. $1 = £0.69). However, such large changes usually only occur over a long period or because of some particular economic problem. In general, the changes are very small, although they can take place quite rapidly. Even then, although the changes may be very slight, their effects can be very large when you consider the amounts of money involved – companies may have many millions of pounds sterling invested in overseas currencies, and a small change in the value of those currencies can produce substantial losses or gains, depending on the whether the value of the currency rises or falls against the pound.

The rate of exchange is expressed as a ratio. It shows the value of one unit of currency in terms of another currency. Thus, one side of the ratio is always "1". Here are some examples of the rate of exchange of the British £:

£1 = 10.55 FF

£1 = 3.15 DM (German marks)

£1 = $1.44

Note that, although it is a ratio, the rate of exchange is usually shown as "x = y", rather than "x : y".

It is this ratio which determines how much of one currency can be bought with another currency, or how much a particular amount of one currency is worth in another.

Before we look at the relevant calculations involved in converting one currency to another, there are two further points to note about the rate of exchange.

- For any relationship between two different currencies, the rate of exchange can be expressed in two ways – how much one unit of one currency is worth in terms of the other, and vice versa. Thus, the rate of exchange between the £ and the DM could be shown as:

  £1 = 3.15 DM or

  1 DM = £0.32

- The rate of exchange is usually quoted in two forms – a selling price and a buying price. The selling price determines the amount of foreign currency you will be sold for your pounds, and the buying price determines the amount of pounds you will receive in return for your foreign currency. (The difference between the two enables the bank or other agency undertaking the exchange to make a profit.) As an example of the two different prices, the rates quoted might be:

		Buying rate	*Selling rate*
£1	=	10.6 FF	10.5 FF
£1	=	3.16 DM	3.14 DM
£1	=	$1.43	$1.45

Currency Conversion

To find the value of a sum of money in one currency in terms of another currency, we could develop a simple formula using the rate of exchange, as follows

let e = rate of exchange expressed as the number of units of a second currency equivalent to one unit of a first currency (as in $1.44 = £1)

x = amount in first currency

y = amount in second currency

To convert a certain amount of the first currency (say, £x) into the second currency (say $y), we multiply the amount of the first currency by the rate of exchange. Thus the formula would be:

$$y = x \times e$$

So, to convert £100 into $ at the rate of £1 = $1.44, we would substitute the following values into the formula:

$$x = 100$$

$$e = 1.44$$

The formula gives us the following calculation:

$$\$y = 100 \times 1.44 = \$144$$

If we wanted to know much a particular sum of the second (say $y) currency would be worth in the first (say £x) currency, we would have to rearrange the above formula to isolate "x". To do this, we simply divide each side of the equation by "e" – the rate of exchange:

$$\frac{y}{e} = \frac{x \times e}{e}$$

$$x = \frac{y}{e}$$

We can now apply these formulae to any given problem.

(a) What is the cost in £ of a bill of exchange in Paris for 1,275 francs if the rate of exchange is £1 = 9.90 FF?

The rate of exchange is expressed as 9.90 units of one currency is equivalent to 1 unit of the other. So, FF is the second currency and £ the first, and "x" is the amount in £s and "y" the amount in FF.

Substituting in the second formula (to find x) we get:

$$£x = £\frac{1,275}{9.90} = £128.79$$

(Note that the actual result of the calculation is £128.78787, but we round up to express it correct to two decimal places as required for currency.)

(b) Find the value of £63.40 in Swiss francs if the rate of exchange is £1 = 3.90 SF

The rate of exchange is expressed as 3.90 units of the second currency (SF) is equivalent to 1 unit of the first (£). So, again "x" is the amount in £s and "y" the amount in SF.

Substituting in the first formula (to find y) we get:

$$y \text{ SF} = 63.40 \times 3.90 = 246.26 \text{ SF}$$

(c) Convert $3,000 to pounds given that £1 = $1.45

The rate of exchange is expressed as 1.45 units of the second currency ($) is equivalent to 1 unit of the first (£). So, again "x" is the amount in £s and "y" the amount in $.

Substituting in the second formula (to find x) we get:

$$£x = £\frac{3,000}{1.45} = £2,068.97$$

(d) Convert 1,000 DM to pounds given that £1 = 1.31 DM

This should be becoming clear to you now.

$$£x = £\frac{1,000}{1.31} = £763.36$$

(e) Find the value of £250 in dollars if the exchange rate is £1 = $1.5

$$\$y = 250 \times 1.5 = £375$$

Practice Questions 1

1. John and Paul ate a meal in a restaurant whilst on holiday in Corfu. The meal for two cost 4,256 drachma. Calculate the cost of the meal in pounds. The exchange rate was 266 Dr to the pound.

2. The Smiths spent a holiday touring in France. While travelling they used 200 litres of petrol which cost 4.7 francs per litre. The exchange rate was 9.8 francs to the pound.

   (a) how much did the petrol cost them in pounds?

   (b) what was the price per litre of the petrol in pence?

3. Before going on holiday to Germany and Austria, the Jones family changed £600 into German marks. While in Germany they spent 824 DM and then changed the remaining marks into Austrian schillings as they crossed the border. The exchange rate was 2.798 DM to the pound and 7.047 Sch to 1 DM. Calculate:

   (a) the number of marks they received;

   (b) the number of schillings they bought.

4. The bank buys francs at 10.05 francs and sells at 9.45 francs.

   (a) Mr Watson changed £100 into French francs for a day trip to France. How many francs did he receive?

   (b) Sadly the excursion was cancelled and so he changed all his francs back into pounds. How much money did he lose because of the cancellation?

Now check your answers with the ones given at the end of the unit.

B. DISCOUNTS AND COMMISSION

You've probably heard the terms "discount" and "commission". What do they mean to you?

Discounts and commissions are a practical use of percentage calculations. Let us look in more detail at their exact meanings and the way in which they are calculated.

Discounts

A discount is a reduction in the selling price of an item, usually offered as an inducement for quantity buying or for prompt payment. The discount is expressed as a percentage of the ***list price*** (the list price is the original price) and what the buyer actually pays is the ***net price*** – the list price minus the discount.

To calculate a discount (D), you use the following rule or formula:

Multiply the list price (or base, B) by the rate of the discount (R) and divide by 100. Thus:

$$D = B \times \frac{R}{100}$$

Consider the following examples.

(a) A three-piece suite was listed as £720, but was sold at a discount of 15%. What is the net price?

$$\text{Discount} = £720 \times \frac{15}{100}$$

$$= £108$$

$$\text{Net price} = £720 - £108 = £612$$

(b) A car was listed at £3,600 but was sold for cash less a discount of 20%. What was the final selling price?

$$\text{Discount} = £3,600 \times \frac{20}{100}$$

$$= £720$$

$$\text{Net Price} = £3,600 - £720$$

$$= £2,800$$

Commission

A commission is the amount of money paid to an agent or broker or salesman for buying or selling goods or services. This is expressed as a percentage of the value of the goods/services traded and is called the rate of commission.

To find the commission, we simply multiply the value of the goods/services traded (known as the principal) by the rate of commission.

Consider the following example.

> Pauline obtained £6,400 of new business for the PR agency for which she works. She is paid commission of 12½%. Calculate the amount due to her.

$$\text{Commission} = 12\tfrac{1}{2}\% \text{ of } £6{,}400$$

$$= £6{,}400 \times \frac{12.5}{100}$$

$$= £6{,}400 \times \frac{25}{200}$$

$$= £800$$

Commissions are also usually charged by banks, travel agents and bureau de changes to cover their costs when carrying out a currency exchange. The commission on changing cash is commonly £3 or 2% whichever is larger.

When going on holiday, most people take a limited amount of cash in the currency of each country to be visited and take the remainder that they will need in the more secure form of travellers cheques. The commission charged on travellers cheques is commonly £3 or 1% whichever is larger.

In these cases, the person exchanging the currency or buying travellers cheques is charged either:

- a ***fixed*** cost – £3 per transaction; or
- a ***variable*** cost – the value of 1% or 2% of the amount of the transaction.

We shall consider fixed and variable elements in more detail later, but consider the following example in relation to currency exchanges.

> Using the rates of commission quoted above, and given that £1 = $1.47, how many dollars would you get when converting:
>
> (a) £100 to American dollars in cash?
>
> (b) £200 to American dollars in cash and in travellers cheques?
>
> In each case, we would need to work out the commission at both the fixed and the variable rate, and then apply whichever is the larger.
>
> (a) The commission payable on converting £100 into $ cash is either £3 or £2 (2% of £100). The fixed rate is the larger so that would be applied:
>
> $$£100 - £3 = £97$$
>
> $$£97 \times 1.47 = \$142.50$$
>
> (b) The commission payable on converting £200 into $ cash is either £3 or £4 (2% of £200). The variable rate is the larger so that would be applied:
>
> $$£200 - £4 = £196$$
>
> $$£196 \times 1.47 = \$288.12$$
>
> However, the commission payable on converting £200 into $ travellers cheques is either £3 or £2 (1% of £200). The fixed rate is the larger so that would be applied:
>
> $$£200 - £3 = £197$$
>
> $$£197 \times 1.47 = \$289.59$$

Practice Questions 2

1. What is the price paid for a £310 washing machine with a discount of 10%?

2. What is the price paid for a £7,000 car with a discount of 15%?

3. An estate agent charges a commission of 1.5% of the value of each house he sells. How much commission is earned by selling a house for £104,000.00?

4. Calculate the commission earned by a shop assistant who sold goods to the value of £824 if the rate of commission is 3%?

5. A car salesman is paid 2.5% commission on his weekly sales over £6,000. In one particular week, he sold two cars for £7,520 and £10,640. What was his commission for that week?

6. On a holiday touring Germany, Switzerland and Austria, a family of four decided to take the equivalent of £900 abroad. The travel agent charges commission on each currency exchange at the rate of £3 or 2% (whichever is the larger) for cash and £3 or 1% (whichever is the larger) for travellers cheques.

   (a) They changed £110 into Deutschmarks, £100 into Swiss francs and £120 into Austrian schillings. How much of each currency did they receive after commission was deducted.

   (b) After buying the foreign currencies, they changed as much as possible of the remaining money into travellers cheques. The smallest value of travellers cheque which can be bought is £10.

      (i) How much did they exchange for travellers cheques?

      (ii) How much commission did they pay for the cheques?

   (c) What was the total cost per person of the foreign currencies and travellers cheques?

   Use the following exchange rates:

   £1 = 2.798 Deutschmarks (DM)

   £1 = 2.458 Swiss franc (SF)

   £1 = 19.65 Austrian schillings (AS)

Now check your answers with the ones given at the end of the unit.

C. SIMPLE AND COMPOUND INTEREST

We introduced the subject of interest in the last unit and will now consider it in more detail. Interest calculations involve the application of simple formulae to specific problems.

Interest is what we call the amount of money someone will give you for letting them borrow your money or what you pay for borrowing money from someone else. Hence, you can receive interest on a loan you make to someone and you can also be asked to pay interest on a loan which you take out.

Note that, when you receive interest on savings from a bank or a building society, this is the bank or building society paying interest to you on money which have loaned to them.

Let us consider the position from the point of view of the borrower. It is often the case that individuals or companies wish to spend money which they do not currently have, but feel sure that they will do in the future. Therefore they borrow that money from a lender – a bank, building society, credit card company, etc. – and pay the amount borrowed back to the lender over a particular period. The cost charged by the lender can vary a great deal depending on the method of obtaining credit, the amount borrowed and the time over which the loan is repaid. This cost is then quoted as an annual percentage rate of interest to be charged on the loan (known as the principal).

Usually finance companies quote a particular rate of interest applicable to either loans taken out or savings placed with them, the rate varying according to amount of the loan/savings and also the period of the loan (or the amount of time you guarantee to leave the savings with them).

There are two ways of looking at interest – simple interest and compound interest.

Simple Interest

Simple interest (SI) is calculated on the basis of having a principal amount, say P, in the bank for a number of years, T, with a rate of interest, R. As we have seen, there is a simple formula to work out the amount of interest your money will earn:

$$\text{SI} = \frac{\text{PRT}}{100}$$

This means that the interest is found by multiplying the principal by the rate, then by the time, and then dividing by 100.

(Note that the way in which this formula is expressed is slightly different from that used in the previous unit. It works in exactly the same way, but is stated in a rather easier fashion. Note, too, that the notation is also slightly different. It is often the case that formulae can be expressed in slightly different ways.)

As we have seen, we can also rearrange this formula to allow us to work out any of the variables. Thus:

$$\text{P} = \frac{100\text{SI}}{\text{RT}}$$

$$\text{R\%} = \frac{\text{SI}}{\text{PT}}$$

$$\text{T} = \frac{100\text{SI}}{\text{PR}}$$

Consider the following examples.

(a) John had £16.40 in an account that paid simple interest at a rate of 9%. Calculate how much interest would be paid to John if he kept the money in the account for 5 years.

The principal (P) is £16.40, the rate (R) is 9% and the time (T) is 5 years. So, substituting in the formula, we get:

$$\text{SI} = \frac{16.40 \times 9 \times 5}{100} = £5.90$$

(b) Calculate the length of time taken for £2,000 to earn £270 if invested at 9%.

Here need to work out T where P = £2,000, SI = £270 and R = 9. Substituting in the formula, we get:

$$T = \frac{100 \times 270}{2{,}000 \times 9} = \frac{27{,}000}{18{,}000} = 1.5 \text{ years}$$

Practice Questions 3

1. What is the simple interest on £250 over 3 years at 8.75% interest rate?

2. If you invest £6,750 at 8.5% per annum, how much interest will you earn and how much will you have in your account after 4 years?

3. If you invest £5,000 at 9.25% per annum for 6 months, how much interest will you receive?

4. £10,800 was invested and at the end of each year the interest was withdrawn. After 4 years, a total of £3,240 in interest had been withdrawn. At what annual rate of interest was the money invested?

Now check your answers with the ones given at the end of the unit.

Compound Interest

Let us now look at how compound interest differs from simple interest.

If an amount of money is invested and the interest is added to the investment, then the principal increases and, hence, the interest earned gets bigger each year.

Suppose you begin with £1,000 and the interest is 8% per year. The interest in the first year is 8% of £1,000 which is £80. Adding this to the original principal, the amount invested at the end of the first year is £1,080.

Now, in the second year, the interest is 8% of £1,080 which is £86.40. The amount accrued at the end of the second year is £1,166.50.

You can continue working out the amounts in this way. However, to find the amount at the end of any year, you are finding 100% + 8% = 108% of the amount at the beginning of the year. So you can multiply by 1.08 each time.

There is a standard formula which can be used to calculate compound interest, as follows:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where: A = Accrued amount

P = Original principal

r = Rate of interest (for a particular time period, usually annual)

n = Number of time periods

Consider the following examples:

(a) £3,000 is invested in an account paying 12% compound interest per year. Find the value of the investment after 3 years.

Working this out year by year, we get:

Investment at the beginning of year 1 = £3,000

Interest at 12% (of £3,000) = £360

Value of the investment after 1 year = £3,000 + £360 = £3,360

Interest at 12% (of £3,360) = £403.20

Value of investment after 2 years = £3,360 + £403.20 = £3,763.20

Interest at 12% (of £3,763.20) = £451.58

Value of the investment after 3 years = £3,763.20 + £451.58 = £4,214.78

Alternatively, we could use the formula:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where: P = £3,000

r = 12%

n = 3

Substituting these values in the formula gives:

$$\begin{aligned} A &= P\left(1 + \frac{r}{100}\right)^n \\ &= £3{,}000\left(1 + \frac{12}{100}\right)^3 \\ &= £3{,}000 \times 1.12^3 \\ &= £3{,}000 \times 1.404928 \\ &= £4{,}214.78 \text{ (to the nearest p)} \end{aligned}$$

(b) £2000 is invested in a high interest account for one and a half years. The interest rate is 10.5% per annum, and it is paid into the account every six months. Calculate the value of the investment after this time and the amount of interest earned.

The rate of interest for 1 year = 10.5%

Therefore, the rate of interest for 6 months = 5.25%

Working this out by each six month period, we get:

Investment at the beginning of year 1 = £2,000

Interest at 5.25% (of £2,000) = £105

Value of the investment after 6 months = £2,000 + £105 = £2,105

Interest at 5.25% (of £2,105) = £

Value of investment after one year = £2,105 + £110.51 = £2,215.51

Interest at 5.25% (of £2,215.51) = £116.31

Value of the investment after 1½ years = £2,215.51 + £116.31 = £2,331.82

Total compound interest earned = £2,331.82 – £2,000 = £331.82

Note that at each stage the amount of interest has been rounded to the nearest penny

Alternatively, we could use the formula:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where: P = £2,000

r = 5.25%

n = 3

Substituting these values in the formula gives:

$$\begin{aligned} A &= P\left(1+\frac{r}{100}\right)^n \\ &= £2{,}000\left(1+\frac{5.25}{100}\right)^3 \\ &= £2{,}000 \times 1.0525^3 \\ &= £2{,}000 \times 1.1659134 \\ &= £2{,}331.83 \text{ (to the nearest p)} \end{aligned}$$

Total compound interest earned = £2,331.83 – £2,000 = £331.83

Practice Questions 4

1. £6,000 is invested at 10% per annum compound interest which is paid annually. How much is in the account after 3 years?

2. £2,000 is invested at 10% per annum payable every 6 months. How much is in the account at the end of 1½ years?

3. One building society offers an interest rate of 8.5% per annum payable half-yearly. A second offers 8.75% payable annually. If you had £2,000 to invest, which savings account should you choose?

Now check your answers with the ones given at the end of the unit.

D. DEPRECIATION

Depreciation is a measure of the wearing out, consumption or other loss of value of an asset arising from use, passage of time, or obsolescence through technology or market changes. For example, a car loses value over time and is worth less each year.

This is an important concept in business where certain types of asset need to be accurately valued in order to help determine the overall value of the business. The amount by which an asset is depreciated each year may also have important consequences.

There are several methods for calculating depreciation, but the most commonly used are the ***straight-line method*** (using fixed instalments) and the ***reducing-balance method*** (applying a fixed percentage). It is acceptable to use different methods for different categories of asset, as long as the method chosen is applied consistently from year to year.

Straight-Line Depreciation

In the straight-line method, depreciation is calculated by dividing the cost of the asset by the number of years the asset is expected to be used in the business. We can express this as a formula as follows:

$$\text{Annual depreciation} = \frac{\text{Cost of asset}}{\text{Useful life}}$$

Thus, over the period of years selected, the value of the asset will be reduced to zero.

In the case of certain types of asset, such as cars, it may be anticipated that the asset will be used for, say, three years and then sold. In this case, the amount to be depreciated is the difference between the cost of the asset and its anticipated value at the end of its useful life, divided by the number of years over which it will be used. The formula would then be:

$$\text{Annual depreciation} = \frac{(\text{Cost of asset}) - (\text{Value at end of useful life})}{\text{Useful life}}$$

Consider the following examples.

(a) A computer costing £1,500 will have a useful life of 3 years. Show the process of depreciation.

$$\text{Annual depreciation} = \frac{£1{,}500}{3} = £500$$

End of year 1: Value = £1,500 – £500 = £1,000

End of year 2: Value = £1,000 – £500 = £500

End of year 3: Value = £500 – £500 = £0

(b) A machine cost £4,800. How much will it be worth after 3 years if it depreciates at £600 per year, and how many years will it be before the machine is valueless?

Depreciation after 3 years will be 3 × £600 = £1,800

The value of the machine will be £4,800 – £1,800 = £3,000

If the annual depreciation is £600, the period over which the machine is depreciated to zero will be:

£4,800 ÷ £600 = 8 years

Reducing Balance Depreciation

In the reducing-balance method, depreciation is calculated each year by applying a fixed percentage to the diminishing balance. Unlike the straight-line method, then, the total provision for depreciation will never equal the cost price at the end of the asset's useful life (i.e. the final balance never reaches zero).

The percentage to be applied will depend on the type of asset. For example, assets having long lives, such as buildings, will have a small percentage (say 2½%), but assets which quickly depreciate, such as motor vehicles, will bear a large percentage (say 15%).

Consider the following example.

A company bought a machine for £6000. If it depreciates at 12% per annum what will it be worth after 3 years?

Amount of depreciation in year 1 = 12% of £6,000

$$= £6{,}000 \times \frac{12}{100}$$

$$= £6{,}000 \times 0.12$$

$$= £720$$

Therefore, value of machine at the end of year 1 = £6,000 – £720 = £5,280

Amount of depreciation in year 2 = 12% of £5,280

$$= £5{,}280 \times \frac{12}{100}$$

$$= £5{,}280 \times 0.12$$

$$= £633.60$$

Therefore, value of machine at the end of year 2 = £5,280 – £633.60 = £4,646.40

Amount of depreciation in year 3 = 12% of £4,646.40

$$= £4{,}646.40 \times \frac{12}{100}$$

$$= £4{,}646.40 \times 0.12$$

$$= £557.57$$

Therefore, value of machine at the end of year 3 = £4,646.40 – £557.57 = £4,088.83

This is, however, obviously a time consuming method. A quicker and simpler way is to realise that 12% is 12/100, which is 0.12. If you are reducing the amount by a factor of 0.12 (12%), you are left with 0.88 (88%) of the amount. So, multiplying the amount by 0.88 gives the same result as multiplying the amount by 0.12 and then subtracting this from the original amount.

We can demonstrate this in respect of the above example as follows:

Amount of depreciation in year 1 = 12% of £6,000

$$= £6{,}000 \times \frac{12}{100}$$

$$= £6{,}000 \times 0.12$$

$$= £720$$

Therefore, value of machine at the end of year 1 = £6,000 – £720 = £5,280

This is the same as:

Value of machine at the end of year 1 $= £6{,}000 \times 88\%$

$= £6{,}000 \times 0.88$

$= £5{,}280$

This is both quicker and easier and you are less likely to make a mistake. Repeatedly multiplying by 0.88 will give the amount left each year. Thus:

At the end of the year 2, the value of the machine $= £5{,}280 \text{ x } 0.88$

$= £4{,}646.40$

At the end of the year 3, the value of the machine $= £4{,}646.40 \times 0.88$

$= £4{,}088.83$

We can express this method by the following formula:

$$D = B(1 - i)^n$$

where: D = Depreciated value at the end of the nth time period

B = Original value at beginning of time period

i = Depreciation rate (as a proportion)

n = Number of time periods (normally years)

Practice Questions 5

1. A launderette buys a washing machine which cost £2,000.
   (a) How much will it be worth after 2 years if it depreciates at £400 per year?
   (b) How many years before the machine is valueless?

2. A factory buys a machine for £7,000. If the machine will be valueless after 4 years, what should be the amount of depreciation per year?

3. A small business buys a computer costing £4,500. The rate of depreciation is 20% per annum. What is its value after 3 years?

4. A motorbike is bought second hand for £795. Its value depreciates by 11% per year. For how much could it be sold two years later? (Give your answer to the nearest pound.)

5. Company A bought a fleet of 20 cars for its sales force, each costing £15,250. It decided to write off equal instalments of the capital cost over 5 years. Company B bought an exactly similar fleet, but it decided to write off 20% of the value each year. Calculate the value of the cars to Company A and to Company B at the end of the 3rd year.

Now check your answers with the ones given at the end of the unit.

E. PAY AND TAXATION

Pay and taxation are complex issues and it is not our intention here to explore them in any detail. Rather, we shall concentrate on the principles of their calculation, using only a basic construction of how "pay" is made up and the way in which the taxation system works.

Again, we shall be applying a number of the basic numerical concepts we have considered previously – including the distinction between fixed and variable values, percentages and simple formulae. This is also an area where it is very important to adopt clear layout when working through calculations.

Make Up of Pay

All employees receive a financial reward as payment for their labour. At its most basic, this is paid in the form of either a ***salary*** or ***wages***.

Salaries are generally paid monthly and are based on a fixed annual amount. Thus:

$$\text{Monthly pay} = \frac{\text{Annual salary}}{12}$$

For example, an employee with an annual salary of £18,000 would receive £1,500 every month (£18,000 ÷ £1,500).

Wages are generally paid weekly and are usually based on a fixed rate for working a certain number of hours. For example, many wage earners are required to work a fixed number of hours in a week, and they are paid for these hours at a basic hourly rate.

Thus, an office clerk is paid £4.00 per hour for a 40 hour week. What is his weekly wage?

$$\begin{aligned} \text{Weekly wage} &= \text{Rate of pay} \times \text{Hours worked} \\ &= £4.00 \times 40 \\ &= £160 \end{aligned}$$

An employee can often increase a basic wage by working longer than the basic week – i.e. by doing ***overtime***. A higher hourly rate is usually paid for these additional hours, the most common rates being time and a half (i.e. 1½ times the basic hourly rate) and double time (twice the basic hourly rate).

Consider, for example, a factory worker who works a basic 36 hour week for which she is paid a basic rate of £5.84 per hour. In addition, she works 5 hours overtime at time and a half and 3 hours overtime at double time. Calculate her weekly wage.

Basic pay:	=	36 hours × £5.84	=	£210.24
Overtime @ at time and a half:	=	5 hours × (£5.84 × 1.5)	=	£43.80
Overtime @ at double time:	=	3 hours × (£5.84 × 2)	=	£35.04
Total pay:			=	£289.08

People who are employed as salespersons or representatives, and some shop assistants, are often paid a basic salary or wage plus a percentage of the value of the goods they have sold. In some cases, their basic salary or wage can be quite small, or even non-existent, and the ***commission*** on their sales forms the largest part of their pay.

For example, a salesman earns a basic salary of £8,280 per year plus a monthly commission of 5% on all sales over £5,000. What would be his income for a month in which he sold goods to the value of £9,400?

$$\text{Basic monthly pay} = \frac{\text{Annual salary}}{12}$$

$$= \frac{£8,280}{12}$$

$$= £690$$

He earns commission on (£9,400 – £5,000) worth of sales

Commission = 5% of (£9,400 – £5,000)

= $0.05 \times £4,400$

= £220

Total monthly pay = Basic pay + Commission

= £690 + £220

= £910

Practice Questions 6

1. Calculate the weekly wage of a factory worker if she works 42 hours per week at a rate of pay of £5.25 per hour

2. If an employee's annual salary is £18,750 how much is she paid per month?

3. A basic working week is 36 hours and the weekly wage is £306. What is the basic hourly rate?

4. A trainee mechanic works a basic 37.5 hour week at an hourly rate of £4.90. He is paid overtime at time and half. How much does he earn in a week in which he does 9 hours overtime?

5. An employee's basic wage is £6.20 per hour, and she works a basic 5 day, 40 hour week. If she works overtime during the week, she is paid at time and a half. Overtime worked at the weekend is paid at double time. Calculate her wage for a week when she worked five hours overtime during the week and four hours overtime on Saturday.

6. An insurance representative is paid purely on a commission basis. Commission is paid at 8% on all insurance sold up to a value of £4,000 per week. If the value of insurance sold exceeds £4,000, he is paid a commission of 18% on the excess. Calculate his total pay for the 4 weeks in which his sales were £3,900, £4,500, £5,100 and £2,700.

Now check your answers with the ones given at the end of the unit.

Taxation

The total earnings made up in the above way is known as ***gross pay***. Unfortunately, very few employees actually receive all the money they have earned. Certain amounts of money are deducted from gross pay and the pay the employee actually ends up with is the ***net pay*** or take-home pay.

The two main deductions are:

- Income tax; and
- Pension contributions.

There are often other deductions – some being further national or local taxes on pay (such as the UK system of National Insurance) and others being deductions made by the employer for work specific reasons (such as repayments of a loan for a car or for a public transport season ticket).

Income tax is a major source of finance for the government and is used to fund all types of public expenditure. The tax is a percentage of gross pay, based on the amount a person earns in a tax year that begins on 6th April and ends on 5th April the following year. The % rate at which the tax is applied varies according to the level of income. The amount of gross income on which tax is payable (known as ***taxable pay***) may be affected by the provision of certain allowances within the tax system and also by certain elements of pay or deductions being exempted from tax (***non-taxable items***). The rates of tax applied to particular bands of income, and the levels of allowances, are decided each year by the government as part of the Budget.

Most people have income tax deducted from their pay before they receive it, by their employer, who then pays the tax to the government. This method of paying income tax is called PAYE (Pay As You Earn).

The UK system divides taxable pay into three bands of income. For simplicity, we shall consider the bands and the rate of tax applied to each to be as follows:

Annual taxable income	Tax rate
0 – £2,400	10%
£2,401 – £28,000	22%
£28,001 and above	40%

This means that, for a person with a total taxable pay of £15,000 in a year, the total tax liability would be:

£2,400 × 10%	=	£240
(£15,000 – £2,400) × 22%	=	£2,772
		£3,012

These annual tax bands can be converted into monthly or weekly amounts for calculating tax on a PAYE basis and deducting the correct amount from an employees gross pay:

Tax rate	Annual taxable income	Monthly taxable income	Weekly taxable income
10%	0 – £2,400	0 – £200	0 – £46
22%	£2,401 – £28,000	£201 – £2,333	£47 – £538
40%	£28,001 and above	£2,334 and above	£539 and above

There are many possible allowances and non-taxable items which affect the level of taxable pay. Here we shall consider just two.

- ***Personal allowances*** – the amount of money an individual is allowed to earn each year before he/she starts to pay tax. This amount (sometimes called free-pay) depends on the individual's circumstances – for example, whether they are single or married, number of children, etc. If your personal allowances are greater than your actual pay, then you would pay no income tax at all.

  The level of personal allowances are generally set as an annual amount, but this can be converted into monthly or weekly amounts of "free-pay".

- ***Pensions contributions*** – the amounts paid by an individual into a pensions scheme, such as an employer's occupational pension scheme, are not taxable. Therefore, pensions contributions, where they are deducted from pay by the employer, may also be deducted from gross pay before calculating tax. (Note that this does not necessarily apply to any other types of deductions from pay. Where pensions contributions are not paid through the employer, an adjustment may be made to a person's personal allowances to allow for them.)

Now we can develop a framework for calculating the amount of tax payable annually, using the tax bands set out above.

Gross taxable pay	=	Gross pay − Non-taxable items
Net taxable pay	=	Gross taxable pay − Personal allowances
Tax liability	=	First £2,400 of Net taxable pay × 10%
		plus next £25,600 of Net taxable pay × 22%
		plus remainder of Net taxable pay × 40%

Consider the following examples, using the bands and tax rates set out above.

(a) How much annual tax will be paid by someone who earns £18,600 per annum and has personal allowances of £5,800? What is the monthly tax deduction?

There are no non-taxable items specified, so:

Gross taxable pay	=	Gross pay − Non-taxable items		
	=	£18,600		
Net taxable pay	=	Gross taxable pay − Personal allowances		
	=	£18,600 − £5,800		
	=	£12,800		
Tax liability	=	£2,400 × 10%	=	£240
		+ (£18,600 − £2,400) × 22%	=	£3,564
			=	£3,804

The monthly income tax deducted will be £3,804 divided by 12 = £317

(b) How much annual tax will be paid by someone who earns £43,000 per annum, has personal allowances of £4,400 and pays pension contributions through his employer at a rate of 5% of his annual salary.

Here, the pension contributions are non-taxable, so:

Gross taxable pay	=	£43,000 − (£43,000 × 5%)
	=	£40,850

$$\begin{aligned}
\text{Net taxable pay} &= £40{,}850 - £4{,}400 \\
&= £36{,}450 \\
\text{Tax liability} &= £2{,}400 \times 10\% &&= £240 \\
&\quad + £25{,}600 \times 22\% &&= £5{,}632 \\
&\quad + (£36{,}450 - £28{,}000) \times 40\% &&= £3{,}380 \\
& &&= £9{,}252
\end{aligned}$$

Note that taxation questions are often set without reference to income bands with different taxation rates applying, or without reference to deductions from gross pay or personal allowances. Always answer the question using the information supplied. For example:

How much tax will be paid each month by someone who earns £24,000 per annum if the tax rate is 25%?

$$\text{Net taxable pay per month, based on the information supplied} = \frac{£24{,}000}{12} = £2{,}000$$

$$\text{Tax liability} = £2{,}000 \times 25\% = £500 \text{ per month}$$

Practice Questions 7

1. An employee is paid a salary of £24,492 per annum which he receives in 12 monthly payments. He has personal allowances amounting to £4,800. If tax is payable at the rate of 24%, calculate his monthly pay.

2. If, in the next Budget, the Chancellor changes the rate of income tax from 24% to 23%, but wages remain the same, how much less tax will be paid by the employee in question 1 above.

3. Someone whose personal tax allowance is £4,800 is paid a weekly wage of £80. Would this person have to pay tax?

4. An employee has a gross income of £23,000 and personal allowances of £5,000. She pays £3,600 tax. What is the rate of income tax?

5. A weekly paid worker receives a basic wage of £5.50 per hour for a 37.5 hour week. Overtime is paid at a rate of time and a half for the first five hours, and at double time for any additional hours worked.

   She also belongs to an occupational pension scheme, making contributions at a rate of 6% of basic earnings only. As a single mother, she has a personal taxation allowance of £5,850.

   Calculate her net pay in a week when she works 48 hours if the tax bands and rates are as set out above.

Now check your answers with the ones given at the end of the unit.

ANSWERS TO PRACTICE QUESTIONS

Practice Questions 1

1. £16.00
2. (a) £95.92
   (b) 48p
3. (a) 1,678.80 DM
   (b) 6,023.78 Sch
4. (a) 945 FF
   (b) £5.97

Practice Questions 2

1. £31
2. £5,950
3. £1,560
4. £24.72
5. £304
6. (a) They would be charged £3 for each cash transaction. Therefore, they would be changing:

   £107 into DM = 299.39 DM;

   £97 into SF = 238.43 SF

   £117 into AS = 2299.05 AS

   (b) (i) £570

   (ii) £5.70

   (c) Total cost is the £900 worth of foreign currency plus the commission of £9 for the cash transactions and £5.70 for the travellers cheques = £914.70. Spread between the four members of the family, cost per person is £228.68.

Practice Questions 3

1. £65.63
2. £2,295
3. £9,045
4. 7.5%

Practice Questions 4

1. £7,986
2. £2,315.25
3. To compare the effects of the two rates, work out the accrued amount after one year in each case:

   For the first building society, the rate of 8.5% payable six monthly equates to 4.25% as a six month rate. Thus, using the formula for calculating the accrued amount over two six month periods is:

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$= £2{,}000\left(1+\frac{4.25}{100}\right)^2$$

$$= £2{,}000 \times 1.0425^2$$

$$= £2{,}000 \times 1.1659134$$

$$= £2{,}173.61 \text{ (to the nearest p)}$$

   The calculation for the second building society rate of 8.75% paid annually is:

$$= £2{,}000\left(1+\frac{8.75}{100}\right)^1$$

$$= £2{,}000 \times 1.0875$$

$$= £2{,}175.00$$

   Therefore, the second building society, with the rate of 8.75% paid annually, is the better investment.

Practice Questions 5

1. (a) £1,200

   (b) 5 years
2. £1,750
3. The value at the end of each year is 80% of the value at the beginning of the year. Thus:

   Value at the end of year 1 = £4,500 × 80% = £4,000

   Value at the end of year 2 = £4,000 × 80% = £3,200

   Value at the end of year 3 = £3,200 × 80% = £2,560
4. The value at the end of each year is 89% of the value at the beginning of the year. Thus:

   Value at the end of year 1 = £795 × 89% = £707.55

   Value at the end of year 2 = £707.55 × 89% = £629.72 = £630 to the nearest pound

5. Total cost of the fleets to both companies $= £15{,}250 \times 20 = £305{,}000$

Company A:

$$\text{Annual amount of depreciation} = \frac{£305{,}000}{5} = £61{,}000$$

After 3 years, total depreciation $= £61{,}000 \times 3 = £183{,}000$

Value of the fleet after 3 years $= £305{,}000 - £183{,}000 = £122{,}000$

Company B

The value at the end of each year is 80% of the value at the beginning of the year. Thus:

Value at the end of year 1 $= £305{,}000 \times 80\% = £244{,}000$

Value at the end of year 2 $= £244{,}000 \times 80\% = £195{,}200$

Value at the end of year 3 $= £195{,}200 \times 80\% = £156{,}160$

Practice Questions 6

1. £220.50
2. £1,562.50
3. £8.50
4. £249.90
5. £344.10
6. £1,456

Practice Questions 7

1. £1,247.16 per month
2. £16.41 per month
3. No
4. 20%
5. Gross pay is made up as follows:

37.5 hours at the basic rate of £5.50 per hour	=	£206.25
5 hours at (£5.50 × 1.5) per hour	=	41.25
5.5 hours at (£5.50 × 2) per hour	=	60.50
Gross pay	=	£308.00

Non-taxable items are pensions contributions amounting to:

$£206.25 \times 6\% = £12.38$

Gross taxable pay = Gross pay – Non-taxable items

= £308.00 – £12.38

= £295.62

$$\text{Net taxable pay} = \text{Gross taxable pay} - \text{Weekly personal allowances}$$

$$= £295.62 - \frac{£5{,}850}{52}$$

$$= £295.62 - £54.81$$

$$= £240.81$$

Tax liability is based on the weekly income tax bands.

$$\text{Tax liability} = £46 \times 10\% = £4.60$$

$$+ (£240.81 - £46) \times 22\% = £42.86$$

$$= £47.46$$

$$\text{Net pay} = \text{Gross pay} - \text{Pension contribution} - \text{Tax}$$

$$= £308.00 - £12.38 - £47.46$$

$$= £248.16$$

Study Unit 4

Simultaneous and Quadratic Equations

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INTRODUCTION

In Unit 2, we introduced the concept of equations and looked at the basic principles of solving them – i.e. finding out the value of the unknown quantity. In that unit, we considered simple equations where there was only one unknown quantity. Now we shall take our study of equations further by examining equations where there are more than one unknown quantity – simultaneous equations.

We shall also consider the distinction between linear and quadratic equations, a difference which will be developed in the next unit when we look at drawing graphs to represent equations.

Before we start, though, we should recap the key principles already examined in relation to equations. It is essential that you fully understand these before studying this unit, so if you are unsure of any points, go back to Unit 2 and revise the section on equations there.

- An equation is a ***statement of equality between two expressions***. It has two sides and they must always be equal.
- Solving an equation means finding the value of the unknown quantity (or quantities) in the equation – the variable – by using the other numbers, the constants. To do this, the expressions which form the equation may be manipulated with the objective of isolating the unknown on one side of the equation.
- The golden rule in manipulating equations is ***equality of treatment to both sides of the equation***. What you do to one side of the equation, you must also do to the other. Thus, the same number may be added or subtracted from both sides of the equation without changing its equality, and both sides may be multiplied or divided by the same number without changing its equality.
- Quantities may be transposed – transferred from one side of the equation to the other – by changing their signs. Thus, an added or subtracted term may be transposed if its sign is changed from + to −, or from − to +. Similarly, a multiplier may be transferred from one side of an equation by changing it to the divisor on the other, and a divisor may be transferred from one side by changing it to the multiplier on the other.
- Unknowns may also be transposed, using the same rules, in order to collect all the unknown terms on the same side of the equation.

Objectives

When you have completed this study unit you will be able to:

- use the elimination method to solve simultaneous equations including two unknowns;
- formulate problems as simultaneous equations;
- define a quadratic equation and distinguish it from a linear equation;
- solve quadratic equations by the factorisation method.

A. SIMULTANEOUS EQUATIONS

As we said in the introduction, up until now, when solving equations, we had only one unknown quantity to find. However, in the practical world, many situations will arise when there is more than one unknown.

If an equation involves two unknown quantities, we may find any number of pairs of values to satisfy it. Consider the following example:

$x + y = 7$

This may be satisfied by:

$x = 3$ and $y = 4$

$x = 2\frac{1}{2}$ and $y = 4\frac{1}{2}$

$x = -1.3$ and $y = +8.3$, etc.

To determine a particular pair of unknown values, then, we need more information. One way of getting more information is to have another equation – for example:

$x + y = 7$ and $2x + y = 12$

Now there is only one solution pair: $x = 5$ and $y = 2$.

Similarly, if $x + y = 7$ and $x - y = 2$, there is only one solution: $x = 4\frac{1}{2}$ and $y = 2\frac{1}{2}$.

Therefore, to find two unknowns we need two equations, and to find three unknowns we need three equations, etc. These groups of equations are known as ***simultaneous equations***.

Note, though, that for equations to be simultaneous they must satisfy two conditions – they must be ***consistent*** and ***independent***. For example:

- the equations $x + y = 7$ and $2x + 2y = 5$ are inconsistent – they cannot be true simultaneously, and no pair of values which satisfies one will satisfy the other;
- the equations $3x + 3y = 21$ and $2x + 2y = 14$ are not independent – they are really the same equations, and all pairs of values which satisfy the one will satisfy the other.

Solution by Elimination

If two equations are true simultaneously, any other equation obtained from them will also be true. The method of solution by elimination depends on this fact.

By adding or subtracting suitable multiples of the given equations, we can eliminate one of the unknowns and obtain an equation containing only one unknown. Once we have done that, the resulting equation can be solved using the methods we already know, and then the other unknown may be discovered by substitution in the original equation.

The following examples explain the method. Study them carefully.

Example 1

Consider the following pair of equations:

$3x + y = 11$ *(1)*

$x - y = -3$ *(2)*

We can add the two equations together to derive a third equation which will eliminate y:

$$\begin{array}{rcrcr} 3x & + & y & = & 11 \\ x & - & y & = & -3 \\ \hline 4x & & & = & 8 \end{array}$$

Thus, by adding the two equations, we are left with 4x = 8. Therefore, x = 2

If we now substitute that value for x in *(1)*, so that 3x becomes 6:

$$6 + y = 11$$

$\therefore$ y = 5 (The symbol $\therefore$ means "therefore".)

We can check that the values we have determined for x and y are correct by substituting both in equation *(2):*

$$x - y = 2 - 5 = -3$$

Note that eliminating one of the unknowns (in this case, y) was possible because both the "y"s in the original two equations had a coefficient of 1 – i.e. the coefficients were equal. (The ***coefficient*** is simply the factor or number applied to an algebraic term, so the coefficient of the term 3x is 3.)

If we want to eliminate one of the unknowns by adding or subtracting the two equations, the coefficients of the unknowns must be equal.

Example 2

Consider the following pair of equations:

$$6x + 5y = -6 \qquad (1)$$

$$18x + 7y = 6 \qquad (2)$$

We want one of the unknowns to have equal coefficients, but unlike in Example 1, this is not the case here. However, we can see that the coefficient of x in *(2)* is a multiple of the coefficient of x in *(1).* To make the coefficients equal, we derive a changed version of the same equation by multiplying *(1)* by 3 and then subtract *(2)* from this equation:

Multiplying (6x + 5y = –6) by 3 gives:

$$18x + 15y = -18$$

Note that, providing we apply the same proportionate change to each term in the equation (by multiplying or dividing each by the same number), the equation has not been changed. It is not a new, independent equation, but simply a different way of expressing the same equation.

We can now subtract equation *(2)* from this:

$$\begin{array}{rcrcr} 18x & + & 15y & = & -18 \\ 18x & - & 7y & = & 6 \\ \hline & & 8y & = & -24 \\ \therefore & & y & = & -3 \end{array}$$

To find x, we substitute the value of y = –3 in *(1)* so that 5y becomes –15:

$$6x + (-15) = -6$$

By transposition, we get:

$$6x = -6 + 15 = 9$$

$\therefore$ x = 1.5

It is always good practice to check the values by substitution in the other equation, but we will skip that step from now on.

Example 3

Consider the following pair of equations:

$$13x - 7y = 33 \qquad (1)$$

$$9x + 2y = 16 \qquad (2)$$

This time, there are no equal or multiple coefficients, so there is no obvious unknown term which can be easily eliminated. We can, then, choose which unknown to eliminate and, to avoid large numerical terms, we shall decide to eliminate y.

Where there are no equal or multiple coefficients, we need to change the way in which *both* equations are expressed so that we can make the coefficients of one of the unknowns equal. To do this here, we multiply *(1)* by 2 and *(2)* by 7, so that the coefficient of y in both equations is 14. We can then eliminate y by adding the two equations:

$$\begin{aligned} 26x - 14y &= 66 \\ 63x + 14y &= 112 \\ \hline 89x &= 178 \\ \therefore \quad x &= 2 \end{aligned}$$

To find y, we substitute the value of x = 2 in *(2)* so that 9x becomes 18:

$$18 + 2y = 16$$

By transposition, we get:

$$2y = 16 - 18 = -2$$

$$\therefore \quad y = -1$$

Important note: When working through the processes of transposition and substitution, it is not necessary to write − (−7) and + (−15), etc., but be sure to pay special attention to all calculations involving negative numbers.

Practice Questions 1

Solve the following simultaneous equations by finding values for x and y.

1. $3x + 4y = 1$

   $9x + 5y = 10$

2. $5x - 3y = 14$

   $7x + 5y = 38$

3. $3x - y = 8$

   $x + 5y = 4$

4. $2x - 11y = 23$

   $7x - 8y = 50$

5. $9x + 10y = 12$

$2x + 15y = -5$

6. $x + \frac{2}{3}y = 16$

$\frac{1}{2}x + 2y = 14$

7. $1\frac{1}{2}x - \frac{3}{5}y = 12$

$x + \frac{7}{15}y = 21$

Now check your answers with the ones given at the end of the unit.

Now we shall look at some harder examples!

(a) Where the unknowns are denominators

Consider the following equations:

$$\frac{4}{x} - \frac{3}{y} = 4 \qquad \textit{(1)}$$

$$\frac{3}{x} - \frac{7}{y} = 2 \qquad \textit{(2)}$$

Here we can treat $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns and then solve the problem in the usual way.

To eliminate $\frac{1}{y}$, multiply *(1)* by 7 and *(2)* by 3 and then subtract:

$$\frac{28}{x} - \frac{21}{y} = 28 \qquad \textit{(1)}$$

$$\frac{9}{x} - \frac{21}{y} = 6 \qquad \textit{(2)}$$

$$\frac{19}{x} = 22$$

$$\therefore \frac{1}{x} = \frac{22}{19}$$

$$x = \frac{19}{22}$$

Substituting for $\frac{1}{x}$ in *(1)* gives:

$$\frac{4 \times 22}{19} - \frac{3}{y} = 4$$

$$\frac{88}{19} - 4 = \frac{3}{y}$$

$$\frac{3}{y} = \frac{88 - 76}{19} = \frac{12}{19}$$

$$\frac{1}{y} = \frac{4}{19}$$

$$\therefore \quad y = \frac{19}{4} = 4\tfrac{3}{4}$$

Alternatively, you may find it easier to put $\frac{1}{x} = m$ and $\frac{1}{y} = n$, and rewrite the equations as:

$$4m - 3n = 4 \qquad (1)$$

$$3m - 7n = 2 \qquad (2)$$

Solve these in the usual way, but remember to write your answer as x = ? and y = ?, not as m and n.

(b) Forming simultaneous equations from single equations

Equations are sometimes given in the form:

$$x + 5y = 2x - 4y + 3 = 3x + 2y - 4$$

In order to solve this, we need to re-formulate the equation as two simultaneous equations.

Taking the first pair of expressions, we can rearrange the terms to form one equation in the usual form:

$$x + 5y = 2x - 4y + 3$$

$$\therefore \quad -x + 9y = 3$$

Now take the second pair of expressions and rearrange them:

$$2x - 4y + 3 = 3x + 2y - 4$$

$$\therefore \quad 2x - 3y = 4$$

We now have two simultaneous equations as follows:

$$-x + 9y = 3 \qquad (1)$$

$$2x - 3y = 4 \qquad (2)$$

These can be solved in the usual way. To eliminate y, multiply *(2)* by 3 and add to *(1)*:

$$5x = 15$$

$$\therefore \; x = 3$$

$$\therefore \; y = \tfrac{2}{3}$$

Practice Questions 2

Solve the following equations for x and y:

$$\frac{3}{x} + \frac{20}{y} = 13$$

$$\frac{7}{x} + \frac{10}{y} = 2$$

Now check your answers with the ones given at the end of the unit.

B. FORMULATING PROBLEMS AS SIMULTANEOUS EQUATIONS

As we have seen previously, it is very useful to develop mathematical models of real situations to help analyse and solve the problems of those situations. The use of algebraic notation to represent variables in a situation and their combination with numerical constants to form equations are the building blocks of such mathematical models.

We have seen that simple equations can be developed to model a situation and enable us to find the value of one unknown variable where we know the values of all the other variables. In the same way, we can develop models which will enable us to find two or more unknown variables by using simultaneous equations.

The key points about formulating problems as simultaneous equations are the same as doing the same with simple equations:

- Always state clearly what your unknowns represent and give the units you are using.
- Having found the number for which the unknowns stand, give the answer to the question in words.
- Check your answer in the question, not in your own equation.

Consider the following examples and try to form the equations for yourself before looking at the solutions given. Note, too, that unless you are told to find both, or all, of the unknowns, it is not essential to do so, but it is most advisable, as otherwise you have no means of checking your answer.

Example 1

The expenditure of 10 men and 8 boys amounts to £160. If 4 men together spend £18 more than 6 boys, how much does each man and boy spend? (Assume each man spends the same amount and each boy spends the same amount.)

Let: $x = $ expenditure of 1 man in £

$y = $ expenditure of 1 boy in £

Then we can express the information in the situation as equations:

$10x + 8y = 160$ (All terms must be expressed in the same unit, i.e. here pounds.)

and $4x - 6y = 18$

Solving these two equations will answer the problem.

Example 2

In a bag containing black and white balls, half the number of white is equal to a third the number of black, and twice the total number of balls exceeds three times the number of black balls by four. How many balls did the bag contain?

Let: $x = $ number of white balls

$y = $ number of black balls

Expressing the information in the situation as equations, we get:

$$\tfrac{1}{2}x = \tfrac{1}{3}y$$

and $2(x + y) - 3y = 4.$

Example 3

Pay special attention to the following problem. Note how the equations are built up.

A certain number, of two digits, is such that it is diminished by 9 if the digits are reversed. The number itself is six times the sum of its digits. Find the number.

Let: x = the tens figure

y = the units figure

Then the number is 10x + y, and the reversed number is 10y + x.

Expressing the information in the situation as equations, we get:

$10x + y - 9 = 10y + x$ *(1)*

and $10x + y = 6(x + y)$ *(2)*

Rearranging these equations to isolate the constant in *(1)* and simplify both, we get:

for *(1)*: $9x - 9y = 9$

or $x - y = 1$ *(A)*

for *(2)*: $4x = 5y$ *(B)*

If we now multiply *(A)* by 4, we get:

$4x - 4y = 4$

We can then substitute the value of 4x from *(B)* to get:

$5y - 4y = 4$

$y = 4$

Substituting for y in *(A)*, we get:

$x = 5$

The required number is 54.

Practice Questions 3

1. Half the sum of two numbers is 20 and three times their difference is 18. Find the numbers.

2. 6 kg of tea and 11 kg of sugar cost £13.30.

   11 kg of tea and 6 kg of sugar cost £17.30.

   Find the cost of tea and sugar per kilo.

3. The income from advertisements and sales for a college magazine amounted, in a year, to £670. In the following year the income from advertisements was increased by 12½% and the income from sales decreased by $16\frac{2}{3}$%. The total income decreased by £12.50.

   Find the original income from advertisements and sales using a calculation method.

4. A train from London to Birmingham takes 2 hours for the journey. If the average speed of the train were decreased by 5 miles per hour, the train would take 12 minutes longer.

   Find the distance from London to Birmingham and the average speed of the train, using simultaneous equations.

Now check your answers with the ones given at the end of the unit.

C. QUADRATIC EQUATIONS

All the equations we have studied so far have been of a form where all the variables are to the power of 1 and there is no term where x and y are multiplied together – i.e. xy does not feature in the equation. (These types of equation are known as ***linear*** equations since, as we shall see in the next unit, if we draw a graph of the outcome of the equation for all the possible values of the unknowns, the result is a straight line.)

By contrast, a ***quadratic*** equation is one that contains the square of the unknown number, but no higher power. For example, the following are quadratic expressions in x:

$4x^2 + 7x$

$2x^2 - x + 1$

$\frac{1}{2} x^2 - 2.$

We can also have quadratic expressions in x^2. For example, $3x^4 + 2x^2 + 5$ is a quadratic in x^2, since it may be also be written as:

$3(x^2)^2 + 2(x^2) + 5$, or

$3y^2 + 2y + 5$, where y stands for x^2.

Note that these are examples of quadratic ***expressions***. A quadratic ***equation*** is obtained by making such an expression equal to a similar expression or to a number.

One of the features of quadratic equations is that, usually, they cannot be solved definitively. There are generally two possible values for the unknown.

A second feature, reflecting this, is that when we draw a graph of the outcome of a quadratic equation for all the possible values of the unknown, the result is a curve. Again, we shall examine this later.

Solving a Quadratic Equation by Factorisation

If an equation contains two unknown numbers, we cannot determine their values unless we are told something more. For example, if x and y are two quantities and all we are told about them is that their product is 12, we cannot tell either of their values – x may be 3, in which case y must be 4; or x may be $-\frac{1}{2}$, in which case y must be -24, etc. This is true whatever the value of the product, with the one exception that if $xy = 0$, then either $x = 0$ or $y = 0$.

If $x = 0$, y may have any value, and if $y = 0$, x may have any value. They need not both be 0, but one of them *must* be, since the product of two numbers is zero only when one of them is itself zero.

This important fact provides the easiest method of solving a quadratic equation, as is shown in the following examples.

Example 1

Consider the following equation:

$x(x - 3) = 0$

Here, the product of the two factors x and $(x - 3)$ is zero. Therefore, one of the factors must be zero:

either: $x = 0$

or: $(x - 3) = 0$

Further, if $(x - 3) = 0$, then $x = 3$

Thus, the equation is satisfied if $x = 0$, or if $x = 3$.

Having understood the principle of this procedure, we do not need to give it in full every time. The following example simplifies the steps.

Example 2

Consider the following equation:

$$x(x - 5) = 0$$

Either, $x = 0$, or $(x - 5) = 0$

$\therefore \quad x = 0$, or $x = 5$

The important thing to remember is that this applies only to a product ***the value of which is zero***. Therefore, the above method of solving an equation applies only when the ***RHS (right-hand side) of the equation is zero***, while the ***LHS can be expressed in factors***.

Example 3

Consider the following equation:

$$(x - 5)(x - 2) = 4$$

We must rearrange this so that the RHS is zero. First of all, we have to expand the product of the factors:

$$x^2 - 7x + 10 = 4$$

Then we rearrange the equation so that the RHS is zero:

$$x^2 - 7x + 6 = 0$$

Now we have to re-express the LHS in factors:

$$(x - 6)(x - 1) = 0$$

Now we can say that either $(x - 6) = 0$, or $(x - 1) = 0$.

$\therefore \quad x = 6$, or $x = 1$

Check this in the original equation just to justify to ourselves that the method works:

$$(x - 5)(x - 2) = 4$$

If $x = 6$:

$$(6 - 5)(6 - 2) = 1 \times 4 = 4$$

If $x = 1$:

$$(1 - 5)(1 - 2) = (-4)(-1) = +4$$

Practice Questions 4

Solve the following equations:

1. $x(x - 1) = 0$

2. $(x - 1)(x + 2) = 0$

3. $3x(x + 1) = 0$

4. $4x^2 = 0$

5. $(2x - 1)(x + 3) = 0$

6. $(3x - 2)(2x + 3) = 0$

Now check your answers with the ones given at the end of the unit.

You should have found these questions very simple and you may even have been able to work them out in your head. However, no matter how simple an equation is, always copy it down first exactly as it is set. Then, if you are sure of the working, you may give the answer at once, but do not just guess.

On the other hand, you may prefer to omit the "either ... or ..." step. If you do this, remember to change the sign of the number as you transfer it to the RHS of the equation. For example:

$(4x - 3)(2x + 5) = 0$

$\therefore \quad 4x = 3, \quad \text{or } 2x = -5$

$\therefore \quad x = \frac{3}{4}, \quad \text{or } x = -2\frac{1}{2}$

One of the reasons that the questions above were so easy is that the LHS of the equations were already expressed in factor terms. Commonly, though, equations are not so simple – the factors may not already have been found, or the terms are not already collected on one side of the operation. You will, therefore, have to work as follows:

- Collect all the terms on the LHS, so that the RHS is zero.
- Factorise the LHS expression – the product of the factors is then zero.
- Equate each factor in turn to zero, where possible.
- Finally, give your answers as alternative values for x.

Work through the following two examples carefully to make sure that you understand the process thoroughly.

Example 4

Consider the following equation:

$7x^2 + 23x = 60$

First, collect all the terms on the LHS, so that the RHS is zero

$7x^2 + 23x - 60 = 0$

Then factorise the LHS expression:

$(7x - 12)(x + 5) = 0$

Then equate each factor in turn to zero:

either $(7x - 12) = 0$, *or* $(x + 5) = 0$

which may be expressed as:

$\therefore \quad 7x = 12, \quad \text{or } x = -5$

Finally, give your answers as alternative values for x:

$\therefore \quad x = 1\frac{5}{7} \text{ or } -5$

Example 5

Solve $x(5x - 2) = 3(2x - 1)$

Expand the product of the factors:

$$5x^2 - 2x = 6x - 3$$

Collect the terms:

$$5x^2 - 8x + 3 = 0$$

Factorise the LHS:

$$(5x - 3)(x - 1) = 0$$

$\therefore \quad 5x = 3, \text{ or } x = 1$

$\therefore \quad x = \frac{3}{5} \text{ or } 1$

Practice Questions 5

Solve the following equations:

1. $7 - x^2 = 5 - x$
2. $x^2 + 4x + 4 = 0$
3. $x^2 - 10x = 24$
4. $x(x - 2) = 2(2 - x)$
5. $x^2 - 11x + 30 = 0$
6. $2x^2 + 5 = 11x$
7. $10x^2 + 11x + 3 = 0$
8. $(x + 3)(x - 3) = 8x$
9. $(x - 1)^2 + (x + 1)^2 = 34$
10. $(x + 1)^2 - (x - 1)^2 = x^2$
11. $9x^2 = 25$
12. $12x^2 + 12x = 9$

Now check your answers with the ones given at the end of the unit.

Solving Quadratic Equations with No Terms in X

In particular circumstances, there is a quicker and easier method than the factor method of solving a quadratic equation.

Consider the following equation:

$$25x^2 - 36 = 0$$

Using the factor method, this would be solved as follows:

$$(5x - 6)(5x + 6) = 0$$

$$\therefore \quad 5x = 6, \text{ or } 5x = -6$$

$$\therefore \quad x = 1.2 \text{ or } -1.2.$$

There is, however, a simpler method of obtaining this answer and you should use it whenever there is no term in x in the given equation:

$$25x^2 - 36 = 0$$

$$25x^2 = 36$$

$$x^2 = \frac{36}{25}$$

To solve this, take the square root of each side, remembering that there are two square roots numerically equal, but opposite in sign:

$$\sqrt{x^2} = \sqrt{\frac{36}{25}}$$

$$x = \pm\frac{6}{5} \text{ or } \pm 1.2$$

Practice Questions 6

Solve the following equations:

1. $x^2 = 16$

2. $4x^2 = 9$

3. $3x^2 - 27 = 0$

4. $x^2 - a^2 = 0$

5. $9x^2 = 1$

Now check your answers with the ones given at the end of the unit.

Solving Quadratic Equations by Formula

Since the method used above for solving a quadratic equation applies to all cases, even where no real roots can be found, it can be used to establish a formula.

Consider the equation $ax^2 + bx + c = 0$, where a, b and c may have any numerical value.

Rearranging the equation we get:

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

At this point, we need to "complete the square", which means adding or subtracting the expression that is needed in order to be able to factorise the LHS. In this case, it is:

$$\left(\frac{b}{2a}\right)^2$$

Remember that this needs to be added to both sides of the equation to keep it balanced.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}$$

Factorising the LHS we get:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{2a}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is an important formula**.** When using it, always write it down first, then write the values of a, b, c in the particular case, paying great attention to the signs. It can be used to solve any quadratic equation, although you should, use the factor method whenever possible.

Example 1

Consider the following equation:

$$5x^2 - 15x + 11 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where: $a = 5$, $b = -15$, $c = 11$

Therefore:

$$x = \frac{+15 \pm \sqrt{225 - 220}}{10}$$

$$= \frac{15 \pm 2.236}{10}$$

$= \quad 1.72$ or 1.28 (to two decimal places)

Example 2

Consider the following equation:

$$3x^2 + 4x - 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where: $a = 3$, $b = 4$, $c = -5$

Therefore:

$$x = \frac{-4 \pm \sqrt{16 + 60}}{6}$$

$$= \frac{-4 \pm 8.718}{6}$$

$$= \frac{4.718}{6} \text{ or } \frac{-12.718}{6}$$

$= 0.79$ or -2.12 (to two decimal places)

Example 3

Consider the following equation:

$$3 - 2x^2 = 4x$$

First arrange equation with x^2 term positive:

$$2x^2 + 4x - 3 = 0$$

Then apply the formula as usual.

Practice Questions 7

Solve the following by means of the formula:

1. $2x^2 + 13x + 4 = 0$

2. $5x^2 - 3x - 4 = 0$

3. $x^2 - 5x + 2 = 0$

4. $4x^2 + 7x + 1 = 0$

5. $x^2 - x - 1 = 0$

Now check your answers with the ones given at the end of the unit.

Remember that the square of a number is always positive, and that no real square root can be found for a negative number.

Therefore, if the term $b^2 - 4ac$ is negative, no real roots of the equation can be found.

For example:

$$2x^2 + 4x + 5 = 0$$

Applying the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where: $a = 2$, $b = 4$, $c = 5$

Therefore:

$$x = \frac{-4 \pm \sqrt{16 - 40}}{4}$$

$$= \frac{-4 \pm \sqrt{-24}}{4}$$

There are no real roots to this expression.

If $b^2 - 4ac$ is a perfect square, then the square root will be a whole number and you should have solved the equation by the factor method.

ANSWERS TO PRACTICE QUESTIONS

Practice Questions 1

1. $3x + 4y = 1$ (i)

   $9x + 5y = 10$ (ii)

   To eliminate x, multiply (i) by 3 so that the coefficients are equal and then subtract (ii):

$$\begin{aligned} 9x + 12y &= 3 \\ 9x + 5y &= 10 \\ \hline 7y &= -7 \\ \therefore \quad y &= -1 \end{aligned}$$

   Substituting $y = -1$ in (i) we get:

$$\begin{aligned} 3x - 4 &= 1 \\ 3x &= 5 \\ \therefore \quad x &= 1\tfrac{2}{3} \end{aligned}$$

   (Check in (ii): $15 - 5 = 10$)

2. $5x - 3y = 14$ (i)

   $7x + 5y = 38$ (ii)

   To eliminate y, multiply (i) by 5 and (ii) by 3 so that the coefficients are equal and then add:

$$\begin{aligned} 25x - 15y &= 70 \\ 21x + 15y &= 114 \\ \hline 46x &= 184 \\ \therefore \quad x &= 4 \end{aligned}$$

   Substituting $x = 4$ in (i) we get:

$$\begin{aligned} 20 - 3y &= 14 \\ 3y &= 6 \\ \therefore \quad y &= 2 \end{aligned}$$

   (Check in (ii): $28 + 10 = 38$)

The full workings for the next three answers are not given, but the method is indicated.

3. $3x - y = 8$ (i)

   $x + 5y = 4$ (ii)

   To eliminate y, multiply (i) by 5 and add the equations: $16x = 44$, $\therefore x = 2\tfrac{3}{4}$

   or To eliminate x, multiply (ii) by 3 and subtract (i): $6y = 4$, $y = \tfrac{1}{4}$

4. $2x - 11y = 23$ (i)

$7x - 8y = 50$ (ii)

To eliminate y, subtract 8 times (i) from 11 times (ii): $x = 6$

or To eliminate x, subtract 7 times (i) from 2 times (ii): $y = -1$

5. $9x + 10y = 12$ (i)

$2x + 15y = -5$ (ii)

To eliminate y, subtract 2 times (ii) from 3 times (i): $x = 2$

or To eliminate x, subtract 2 times (i) from 9 times (ii): $y = -\frac{3}{5}$

6. $x + \frac{2}{3}y = 16$ (i)

$\frac{1}{2}x + 2y = 14$ (ii)

To eliminate x, subtract 2 times (ii) from (i): $3\frac{1}{3}y = 12$, $10y = 36$, $y = 3\frac{6}{10}$

Substituting for y in (ii):

$$\frac{1}{2}x + 6\frac{6}{5} = 14$$

$$\frac{1}{2}x = 14 - 7\frac{1}{5}$$

$$\frac{1}{2}x = 6\frac{4}{5}$$

$$x = 13\frac{3}{5}$$

7. $1\frac{1}{2}x - \frac{3}{5}y = 12$ (i)

$x + \frac{7}{15}y = 21$ (ii)

Notice that (i) has a factor 3, so divide by this to get a simpler equation:

$$\frac{1}{2}x - \frac{1}{5}y = 4 \qquad \text{(i)}$$

To eliminate x, subtract 2 times (i) from (ii): $(\frac{7}{15} + \frac{2}{5})y = 13$, $\frac{13}{15}y = 13$, $y = 15$

Substituting in (i) for y gives:

$$\frac{1}{2}x - 3 = 4$$

$$\frac{1}{2}x = 7$$

$$x = 14$$

Practice Questions 2

Let $a = \dfrac{1}{x}$, and $b = \dfrac{1}{y}$. Then express the equations using these terms:

$3a + 20b = 13$

$7a - 10b = 2$

Solving these in the normal way gives:

$a = 1$, and $b = ½$

Therefore:

$x = 1$, and $y = 2$

Practice Questions 3

1. *Let:* x = first number

y = second number

Then, if half the sum of two numbers is 20:

$$\tfrac{1}{2}(x + y) = 20$$

or $x + y = 40$ *(i)*

and, if three times their difference is 18:

$$3(x - y) = 18$$

or $x - y = 6$ *(ii)*

Eliminating x by subtraction, we get:

$$2y = 34$$

$$y = 17$$

Substituting for y in *(i)*, we get:

$$x + 17 = 40$$

$$x = 23$$

Therefore, the numbers are 17 and 23.

2. As the units must all be the same, change each amount of money to pence.

Let: x = cost of tea per kg in pence

y = cost of sugar per kg in pence

Then

$6x + 11y = 1{,}330$ *(i)*

$11x + 6y = 1{,}730$ *(ii)*

To eliminate x, subtract 6 times (ii) from 11 times (i) to get:

$$85y = 4{,}250$$

$$y = 50$$

Substituting for y in (i) we get:

$$6x + 550 = 1{,}330$$

$$x = \frac{1{,}330 - 550}{6}$$

$$x = 130$$

Therefore, the cost of tea = £1.30 per kilo, and the cost of sugar = £0.50 per kilo.

3. *Let:* x = the original income from advertisements (in £)

y = the original income from sales (in £)

Thus, in the first year:

$x + y = 670$ *(i)*

In the second year:

$(x + 12\frac{1}{2}\% \, x) + (y - 16\frac{2}{3}\% \, y) = 670 - 12.50$

$(x + \frac{1}{8}x) + (y - \frac{1}{6}y) = 657\frac{1}{2}$

$\frac{9}{8}x + \frac{5}{6}y = 657\frac{1}{2}$

Multiplying this by 24 to eliminate the fractions, we get:

$27x + 20y = 15{,}780$ *(ii)*

To eliminate y from the pair of equations, multiply *(i)* by 20:

$20x + 20y = 13{,}400$

and subtract this from *(ii)* to get:

$7x = 2{,}380$

$x = 340$

Substituting for x in *(i)*:

$340 + y = 670$

$y = 330$

Thus, the original income from:

advertisements = £340

sales = £330

4. Let: x = the average speed (in miles per hour)

y = the distance (in miles)

We can use a formula for the time taken:

$$\frac{\text{distance}}{\text{speed}} = \text{time taken}$$

So at average speed:

$$\frac{y}{x} = 2 \text{ hrs}$$

And at the slower speed:

$$\frac{y}{(x-5)} = 2 \text{ hrs } 12 \text{ mins } = 2\frac{1}{5} \text{ hrs}$$

Rearranging these equations into standard linear form, we get:

$$y = 2x \qquad (i)$$

$$y = 2\tfrac{1}{5}x - 11 \qquad (ii)$$

Subtracting *(ii)* from *(i)* gives:

$$0 = \tfrac{1}{5}x - 11$$

$$11 = \tfrac{1}{5}x$$

$$55 = x$$

$$\therefore \quad x = 55$$

Substituting for x in *(i)*, we get:

$$y = 110$$

Thus:

The average speed for the journey = 55 mph

The distance from London to Birmingham = 110 miles.

Practice Questions 4

1. $x = 0$ or 1
2. $x = 1$ or -2
3. $x = 0$ or -1
4. $x = 0$
5. $x = \tfrac{1}{2}$ or -3
6. $x = \tfrac{2}{3}$ or $1\tfrac{1}{2}$

Practice Questions 5

1. $x = 2$ or -1
2. $x = -2$
3. $x = 12$ or -2
4. $x = 2$ or -2
5. $x = 5$ or 6
6. $x = \tfrac{1}{2}$ or 5
7. $x = -\tfrac{3}{5}$ or $-\tfrac{1}{2}$
8. $x = 9$ or -1
9. $x = 4$ or -4 (written as $x = \pm 4$)
10. $x = 0$ or 4
11. $x = \pm 1\tfrac{2}{3}$
12. $x = -1\tfrac{1}{2}$ or $\tfrac{1}{2}$

Practice Questions 6

1. $x = \pm 4$
2. $x = \pm 1\frac{1}{2}$
3. $x = \pm 3$
4. $x = \pm a$
5. $x = \pm \frac{1}{3}$

Practice Questions 7

1. -0.32, -6.18
2. 1.24, -0.64
3. 4.56, 0.44
4. -0.16, -1.59
5. 1.62, -0.62

Study Unit 5

Introduction to Graphs

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INTRODUCTION

For many problems and issues which fall within the remit of quantitative methods, it is possible to analyse and solve them both mathematically and graphically. This unit introduces the principles and role of graphs in such problems.

There are many different types of graph but here we will only be concerned with linear (straight line) graphs and quadratic (curved line) graphs. Line graphs illustrate the relationships between two variables, showing graphically what happens to one variable when the other changes. For example, many graphs involve time as one of the variables, and thus provide a measure of something that changes with time. Whenever one measurable quantity changes as a result of another quantity changing, it is possible to draw a graph.

You will immediately realise that, since equations are a mathematical expression of the relationship between two variables, it is possible to represent equations by means of graphs. We shall spend some time on this in the unit.

The importance of graphs is that they give a visual picture of the behaviour of the two variables. From this, by knowing how to interpret it, information can very quickly be found. The picture may, for example, indicate a trend in the data from which we might be able to make further predictions. The graph might also be used to approximate a solution to a particular problem.

Graphical approaches to problem solving are often easier to carry out than mathematical ones. However, they are less accurate, since the answers must be read from the graph and hence cannot be more accurate than the thickness of the pencil. For this reason, it is essential that you are very careful when drawing graphs, and we cover some of the key design aspects here. One element of this is the need to use graph paper and you will find it very useful to have a supply to hand as you work through this unit.

Objectives

When you have completed this study unit you will be able to:

- explain the construction of graphs;
- calculate functions of x (or f(x));
- calculate and tabulate the values of y from an equation in x;
- draw graphs from linear and quadratic equations;
- identify the characteristics of a straight line in terms of its equation, and calculate its gradient from its graph or its equation;
- decide, from an equation, which way its graph will slope;
- identify, from looking at their equations, whether two lines are parallel;
- describe a shaded region on a graph in terms of the lines bounding it;
- use graphs to solve problems.

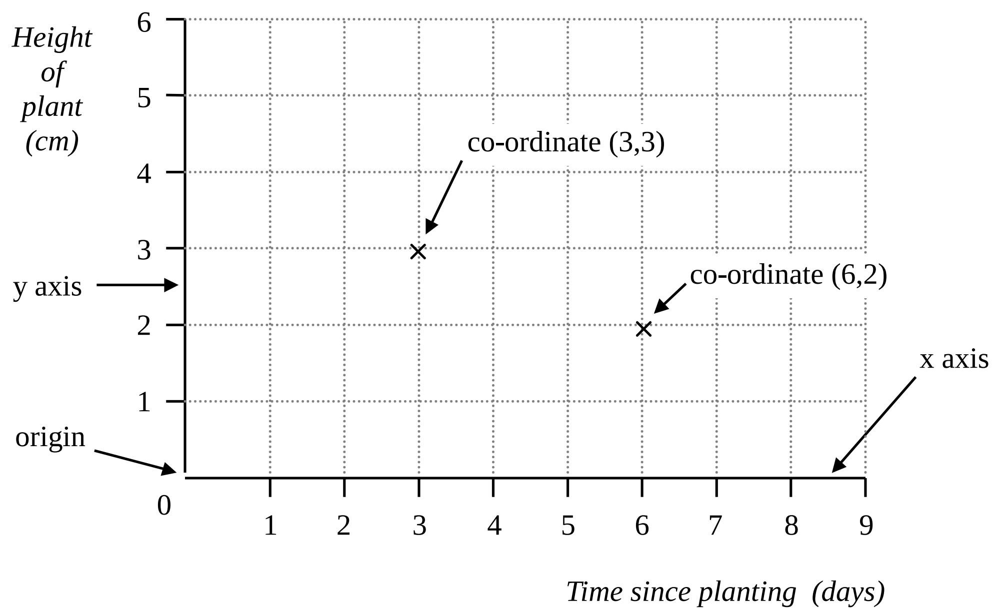
A. BASIC PRINCIPLES OF GRAPHS

A graph is a diagram illustrating one or more mathematical relationships between two variables. The following figures show the relationship between the height of a plant and the time since it was planted. We shall use this to explain some of the basic features of graphs and their construction.

We shall start by considering the framework of the graph.

Framework of a Graph

Figure 5.1: Sample graph framework



A graph has two "***axes***" along which the values of the variables are shown:

- the ***y axis*** is the vertical axis and usually displays the ***dependent*** variable – i.e. the variable which changes according to changes in the other, here the height of the plant which changes in accordance with the time of planting;
- the ***x axis*** is the horizontal axis and usually displays the ***independent*** variable – i.e. the variable which causes change in the other variable, here the time since planting which causes the height of the plant to increase.

The values of the variables are marked along each axis according to a ***scale*** (here, 0 – 9 on the x axis and 0 – 6 on the y axis). Not all values need to be marked, but the dividing marks do need to be equally spaced along the axis and the scale used must be clear to the reader. The scale need not be the same on both axes.

Each axis should be ***titled*** to show clearly what it represents and the units of measurement.

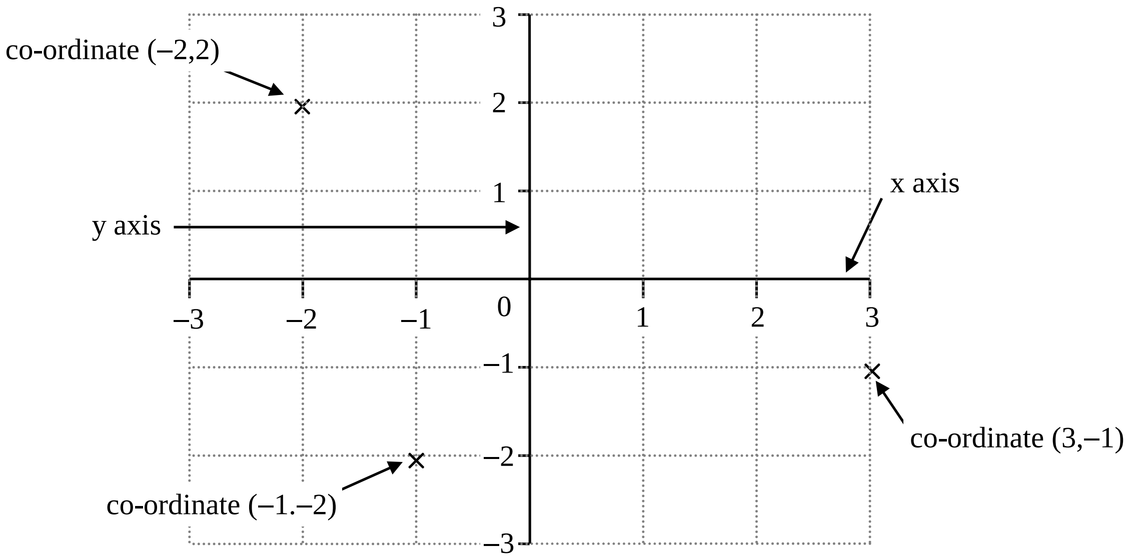
The area within the axes (known as the ***plot area***) is divided into a grid by reference to the scale on the axes. This grid may actually be marked on the paper (as is the case with graph paper, which is why it is much easier to draw graphs using graph paper) or may be invisible. We have shown the main gridlines here as grey, dotted lines.

Particular points within the graph are identified by a pair of numbers called ***co-ordinates*** which fix a point by reference to the scales along the axes. The first co-ordinate number always refers to a value on the scale along the x axis and the second to the scale on the y axis. We have shown two points on the above framework – make sure you understand their referencing.

The point at which the two axes meet is called the ***origin***. This has the co-ordinate reference (0,0).

Finally, here, note that the x and y axes may extend to include negative numbers. In this case, the framework of the graph will look like this:

Figure 5.2: Graph framework including negative scales



Drawing a Graph

When drawing a graph, follow this procedure for constructing the framework.

- Determine the range of values which will need to be shown on each of the axes. You should start by considering the x axis since the range of values for the independent variable will determine the range of values needed for the independent variable on the y axis.
- Draw the axes and mark off the scales along them. The aim is usually to draw as large a graph as possible since this will make it easier to plot the co-ordinates accurately, and also make it easier to read. Remember that the scales must be consistent along each axis, but do not need to be the same on both. If you are using graph paper, use the thick lines as your main divisions. If you are not using graph paper, use a ruler to mark the scale and select easy-to-identify measurements for the divisions (such as one cm for each main unit).
- Give each axis a title, number the main divisions along each scale and give the whole graph a title.

Now you are ready to start plotting the graph itself.

Points on the graph are identified by their co-ordinates. These are the values given or determined for, firstly, the independent variable and, secondly, for the dependent variable. Thus, you may be given the following data about the growth of a plant:

	Height of plant *(cm)*
Day 1	1
Day 3	2
Day 5	3
Day 7	4

The points should be plotted with a small x using a sharp pencil.

To plot the co-ordinate (1,1) you count one unit across (along the x axis) and then count one unit up (along the y axis). Be careful not to get these in the wrong order or when you come to plot the co-ordinate (7,4) you will have plotted the point (4,7), which is somewhere completely different!

(Remember that if the x co-ordinate is negative you count to the left and if the y co-ordinate is negative you count down.)

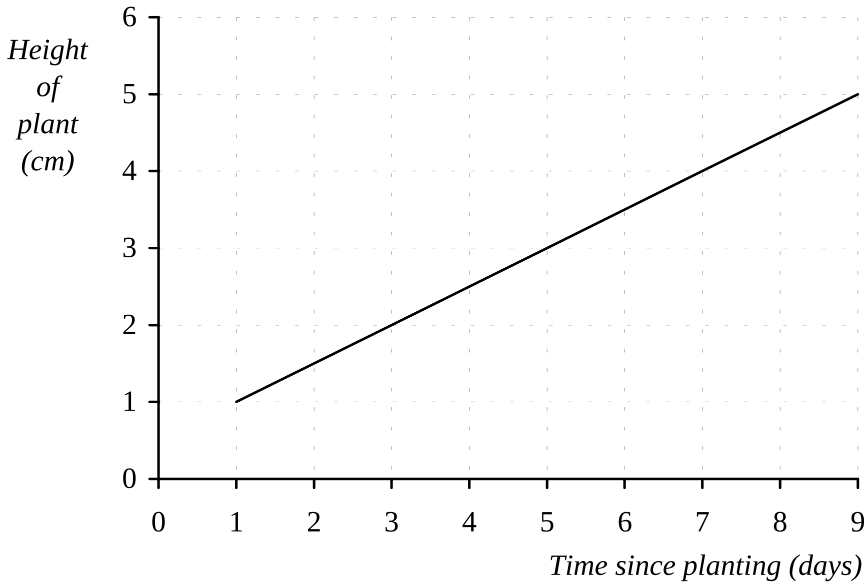
Once you have plotted all the given points, you should join them all up.

If your graph is a linear graph, all the points will be in a straight line and you can join them using a ruler. (In fact, to draw a linear graph, you only need to plot two points which, when joined and extended across the whole scale of the x axis, give the complete graph for all values of the independent variable.)

However, if your graph is a quadratic graph, then the plot line will form a curve. The points should be joined with a smooth curve. The easiest way of doing this is to trace through the points with a pencil, but without touching the paper, to find the shape of the curve. When you are satisfied that the path of the curve is smooth, draw the curve through the points in one movement.

The full graph derived from the above data on the height of a plant and extended over nine days is given in Figure 5.3.

Figure 5.3: Graph of height of plant against time



Now that we have reviewed the basic principles, we shall move on to consider in detail the way in which variables and equations are represented on graphs.

B. GRAPHING THE FUNCTIONS OF A VARIABLE

The statement that something is a “function of x” simply means that the value of that something depends upon the value of a variable quantity, x. It is denoted by f(x). If y = f(x), y will have a different value for each value of x, so that we can obtain a succession of values for y by giving successive values to x.

For example, if y = x + 3, when x is 0, y will be equal to 3; when x = 1, y = 4; when x = 2, y = 5; when x = 3, y = 6, and so on. For convenience, we usually tabulate our values for x and y, as follows:

x	0	1	2	3	4	5
y = x + 3	3	4	5	6	7	8

Let’s look at another example. Suppose you are given the equation $y = x^2 + 3$ and are asked to find the values of y from x = –2 to x = 4.

You could start by making the following table:

x	–2	–1	0	1	2	3	4
x^2	+4	+1	0	+1	+4	+9	+16
+3	+3	+3	+3	+3	+3	+3	+3
$y = x^2 + 3$	7	4	3	4	7	12	19

Alternatively, as this is a simple problem, you could tabulate just the first and the last lines, working the rest in your head as you go along.

We can also find a single function of x by substituting that value directly into the given equation, for example:

$f(x) = x^2 + 2x$

Replace x by 3 and find the value:

$f(3) = 3^2 + 2 \times 3 = 15$

Replace x by –2 and find the value

$f(-2) = (-2)^2 + 2(-2) = 0$

Practice Questions 1

1. Complete the table for $y = 3x - 2$.

x	0	0.5	1	1.5	2	2.5
$y = 3x - 2$						

2. Which one of the following functions could be represented by the table below?

$y = x + 4$

$y = x^2 - 6x + 16$

$y = 2x + 1$

$y = x^3 - 1$

x	3	5	7	9	11
y	7	11	15	19	23

3. $f(x) = 5x - 2$. Find:

(a) $f(2)$

(b) $f(-3)$.

4. $f(x) = x^2 + 2$. Find:

(a) $f(2)$

(b) $f(-3)$

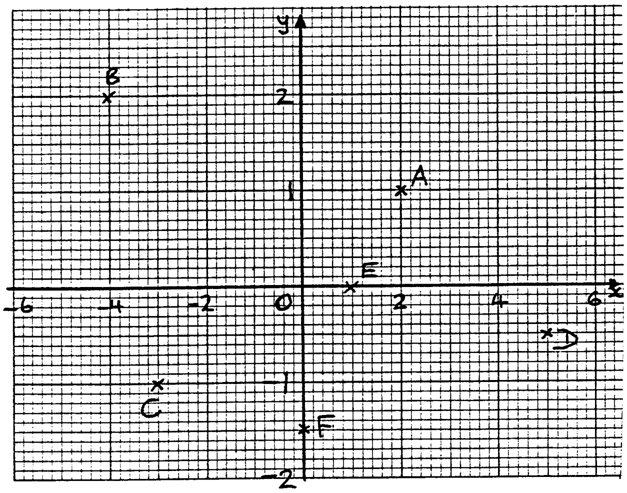
Now check your answers with the ones given at the end of the unit.

Representation

From the tables we see that as the value of x changes, so does the value of y. One way of showing how these values change is by means of a graph.

The values in the table can be expressed as a pair of co-ordinates which may be plotted on a graph. The first figure is the value of x along the scale of the x (horizontal) axis, and the second figure is the value of y along the scale of the y (vertical) axis. For example, in Figure 5.4, A is the point (2, 1), B is (–4, 2), C is (–3, –1), D is $(5, -\frac{1}{2})$, E is (1, 0) and F is $(0, -1\frac{1}{2})$.

Figure 5.4: Co-ordinates for y = f(x)



Ordered Pairs

Another way of describing the varying values of x and y is by means of a set of ***ordered pairs***. This is a very useful and helpful way of noting the points we required in the graphical representation of a function.

The table in question 2. of the first Practice Questions could be written as a set of ordered pairs as follows:

(3, 7)

(5, 11)

(7, 15)

(9, 19)

(11, 23)

These are then the points which we would plot if we were going to draw the graph of the function in question.

When dealing with ordered pairs, you will normally be told the values of x with which you are concerned. Thus, if we had to write down the ordered pairs of the function 2x + 1 for integral (whole number) values of x from −1 to 2 inclusive, we would write (−1, −1), (0, 1), (1, 3), (2, 5); and if the function was $x^2 + 3$ for the same values of x it would be (–1, 4), (0, 3), (1, 4), (2, 7).

C. GRAPHS OF EQUATIONS

In this section we shall consider graphs of two types of equation.

- ***Linear equations*** – equations of the first degree in x and y (i.e. containing no higher power of x than the first, such as y = 2x + 3, y = 5x – 1). As the name implies, the graph of expressions like these will always be a ***straight line***.
- ***Quadratic equations*** – those of the second degree in x and y (i.e. containing the square of the independent variable, but no higher power, such as $y = 2x^2 - x + 1$, $y = ½x^2 + 3$). Such equations always give a ***curved*** graph with a characteristic "U" shape, called a ***parabola***.

We shall also look at graphs from the expression $y = \frac{a}{x}$ where a is a constant, which have a different curved shape – a ***hyperbola***.

Linear Equations

A linear equation is of the form y = mx + c, where m and c are constants. This will always give a straight line.

To draw a line we must know two points on it. It is even better if we know three points, using the third as a check. so, the minimum number of points you ought to plot for a linear graph is three.

If we choose any three values of x we can find the corresponding values of y and this will give us the co-ordinates of three points which lie on the line.

Example 1

Draw the graph of y = 2x – 3.

Suppose we choose x = 0, x = 2 and x = –2.

Taking the given equation, y = 2x = 3:

when x = 0: y = 0 – 3 = –3.

when x = 2: y = (4 – 3) = 1;

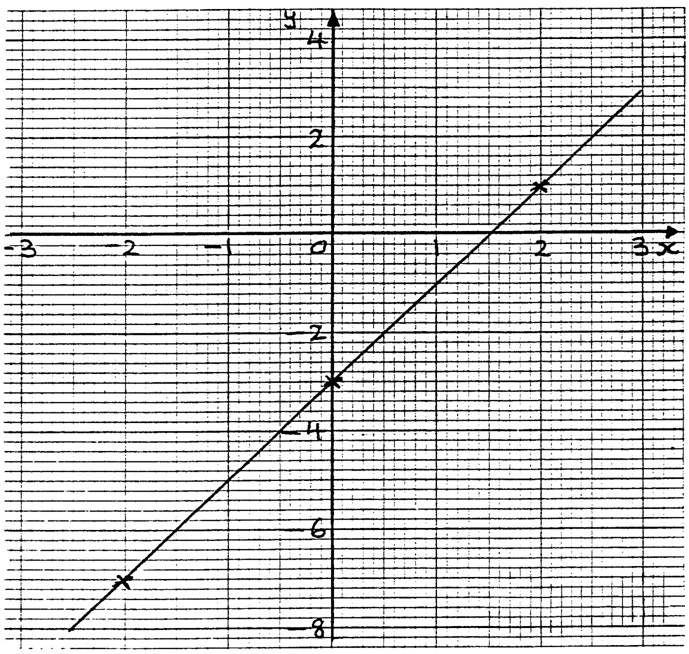
when x = –2: y = (–4 –3) = –7.

This is better tabulated as:

x	0	2	–2
y = 2x – 3	–3	1	–7

We now have three points which we can plot and, on joining them, we have the graph of y = 2x – 3.

Figure 5.5: Graph of y = 2x – 3



Example 2

Try this one yourself. Draw the graph of y = 3x + 2, for values of x from –3 to 2.

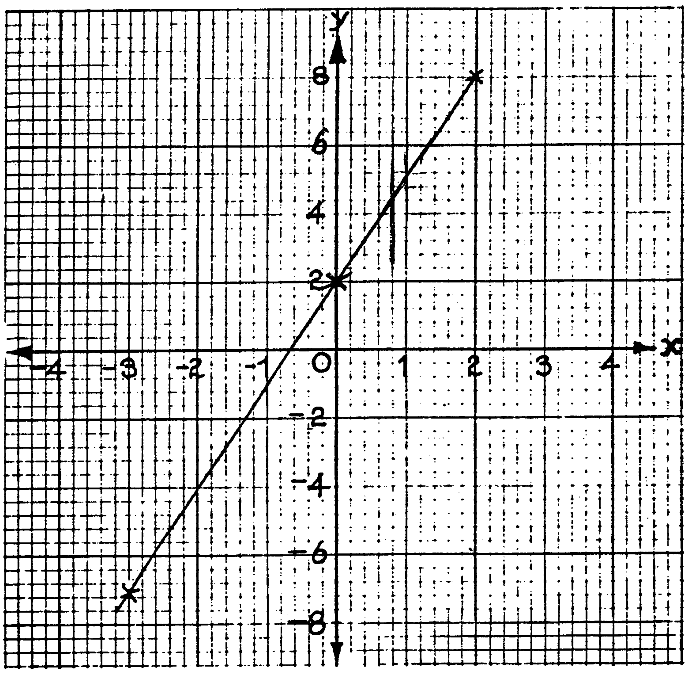
First choose three values of x, say –3, 0 and 2. Then draw up a table of values – fill in the spaces for yourself to find the co-ordinates of the points be plotted.

x	–3	0	2
y = 3x + 2			

You should have found the points: (–3, –7), (0, 2), (2, 8).

Now draw your two axes to cover the range of values for x specified (although it is good practice to extend them beyond this – leaving some "spare" room at each end of each axis), mark the scale, plot your points and join them up. Then compare your result with Figure 5.6.

Figure 5.6: Graph of y = 2x + 2



If you look carefully at Figure 5.6 you will notice that the line y = 2x + 2 meets the y axis at the point where y = 2. This is because the constant "c" in the general equation y = mx + c is always the point where the line passes through the y axis (i.e. where x = 0). A further fact which emerges is that when x goes from 0 to 2 (i.e. increases by two), y rises from 2 to 8 (i.e. increases by four). The change in the value of y for a change of one in the value of x is given in the equation y = mx + c by the constant "c".

We have thus found that in the equation y = mx + c:

the constant c = the point where the line crosses the y axis

the constant m = the slope of the line.

The point at which the line crosses an axis is called the ***intercept***. The y intercept is where the line crosses the y axis and the x intercept is where the line crosses the x axis.

We shall return to these points in some detail later.

Example 3

Suppose we are asked to draw the graph of 3y = 2x + 1, from x = 4 to –2.

Before we can work out the table of values in this case, we must change the equation to the form y = mx + c:

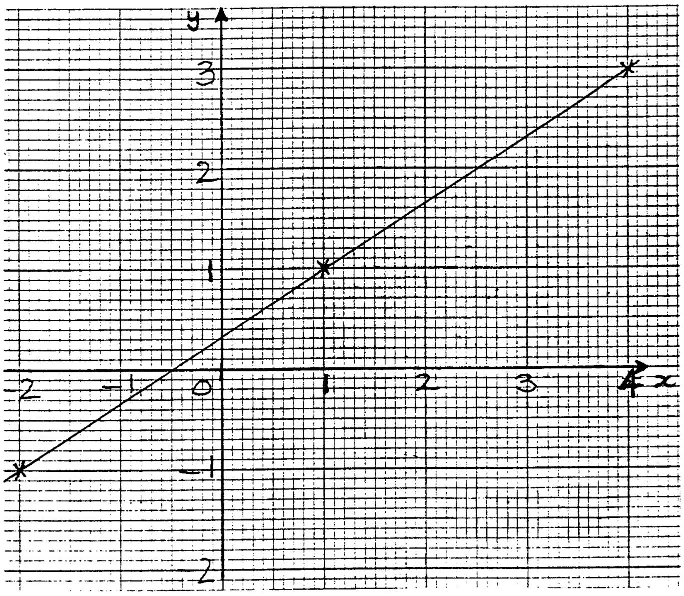
$$y = \frac{2x + 1}{3}$$

Now choose some values of x – let's say 4, 1, –2. Working these out gives us the following table of values:

x	4	1	–2
y	3	1	–1

Now plot these points and join them. Note that, as in the title to Figure 5.7, this is still a graph of the original equation.

Figure 5.7: Graph of 3y = 2x + 1



Example 4

We are not limited to having just one line plotted on a graph. We can draw two or more graphs using the same axes – a very useful method of comparing different equations of the same variables.

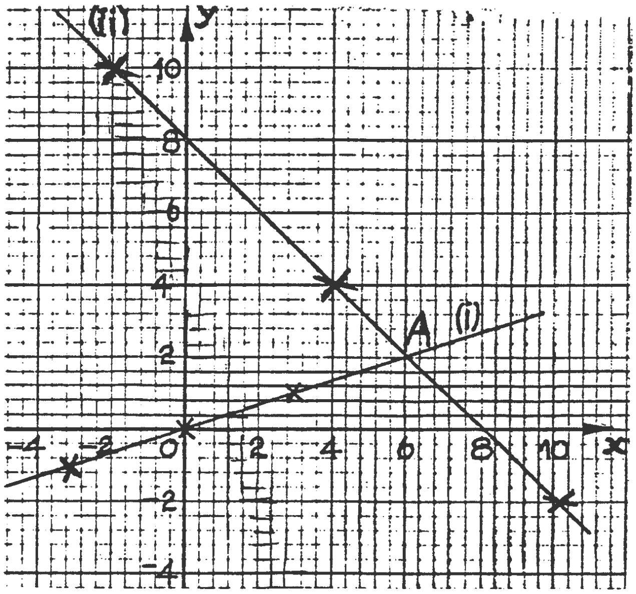
For example, we could draw the graphs of and y = 8 − x on the same axes.

Taking values of x between −4 and 10, we get the following tables of values

For $y = \frac{1}{3}x$	x	–3	0	3
	y	–1	0	1
For y = 8 – x	x	–2	4	10
	y	10	4	–2

The graphs are shown in Figure 5.8.

Figure 5.8: Graphs of $y = \frac{1}{3}x$ ***and*** $y = 8 - x$



You will notice that the lines intersect (i.e. cross) at A (6,2)

Practice Questions 2

1. Draw the graph of y = 5x – 4, from x = –1 to x = 2.

2. Draw the graph of y = 2x + 3, from x = –4 to x = 2.

3. Draw the graph of y = 2 – x, for values of x from –4 to 6.

4. On the same axes, plot the following pairs of points on squared paper, and draw the line through them:

   (i) (3, 0) (0, 6)

   (ii) (–1, 0) (0, –5)

   (iii) (–3, –2) (2, 3)

   (iv) (–4, 3) (–1, 2)

   Write down the co-ordinates of the points where (i) and (ii) each cut the lines (iii) and (iv).

5. For values of x from –3 to 6, draw on the same axes the graphs of:

   (i) y = 5 (ii) x = –5 (iii) y = 5x (iv) $y = -\frac{1}{5}x$

   Describe briefly the resultant lines.

6. For values of x from –3 to 6, draw on the same axes the graphs of:

   (i) $y = 2x$ (ii) $y = 2x + 3$ (iii) $y = 2x - 4$

   Describe briefly the resultant lines and their intercepts.

7. For values of x from –3 to 6, draw on the same axes the graphs of:

   (i) $y = x - 2$ (ii) $3y = 6 - x$

   Identify the point of intersection.

Now check your answers with the ones given at the end of the unit.

Characteristics of straight line graphs

If you consider the graphs we have drawn as examples, and those you drew in the above Practice Questions, we can start to identify a number of characteristics of straight line graphs.

The general equation for a straight line is $y = mx + c$ where m and c are constants which may have any numerical value, including 0. The value of m is known as the ***coefficient*** of x, and plays a very important part in calculations concerning a straight-line graph.

- ***Lines parallel to an axis***

  Consider the case where $m = 0$. This means $y = c$.

  Whatever value x has, y is always equal to c. The graph of $y = c$ must, then, be a line parallel to the x axis at a distance c from it. Note that $y = 0$ is the x axis itself.

  In the same way, $x = c$ is a line parallel to the y axis at a distance c from it. $x = 0$ is the y axis.

- ***Lines through the origin***

  If $c = 0$, then the equation of the line is $y = mx$. When $x = 0$ it is clear that $y = 0$, so the graph of $y = mx$ is a line through the origin (0, 0).

- ***Parallel lines***

  When the coefficient of x is the same for several lines, the lines are parallel. Thus, if m is kept the same and c has different values, we get a set of parallel straight lines.

- ***Direction of slope***

  If the coefficient of x is negative (for example, $y = 4 - 3x$) the line will slope **down** from left to right instead of up, as it does when the coefficient of x is positive. Thus, in $y = mx + c$, if m is positive the line slopes up from left to right; if m is negative the line slopes down.

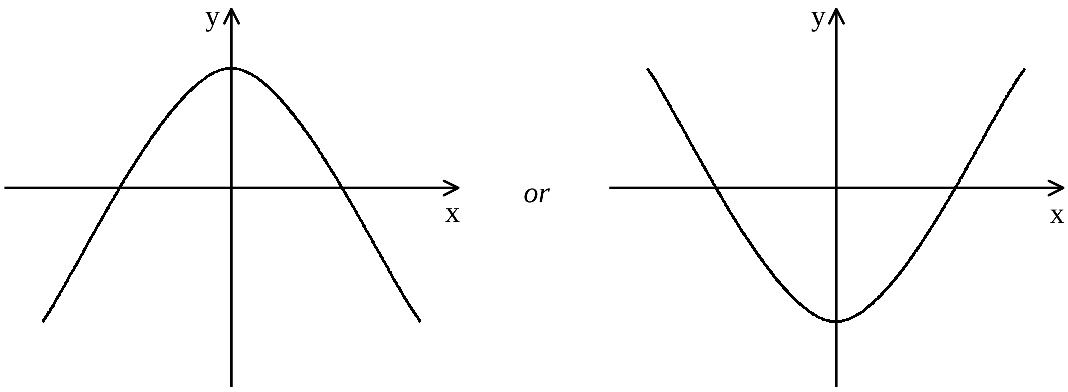
- ***Intercepts***

  The distances from the origin to the point where the line cuts the axes are called the **intercepts on the axes**. They are found by putting $x = 0$ and $y = 0$ in the equation of the graph. In $y = mx + c$, the intercept on the y axis is always at c.

Quadratic Equations

A quadratic equation is of the form $y = ax^2 + bx + c$, where a, b and c are constants. This will always give you a curved graph, and the part that is usually asked for is the part that does a “U” turn.

The general shape of a quadratic equation will be either:



where a is negative | where a is positive

In contrast to linear graphs, in order to draw a quadratic graph accurately, you need as many points as possible – especially around the dip.

The maximum and minimum values of the quadratic equation can be found by careful inspection of the graph.

Example 1

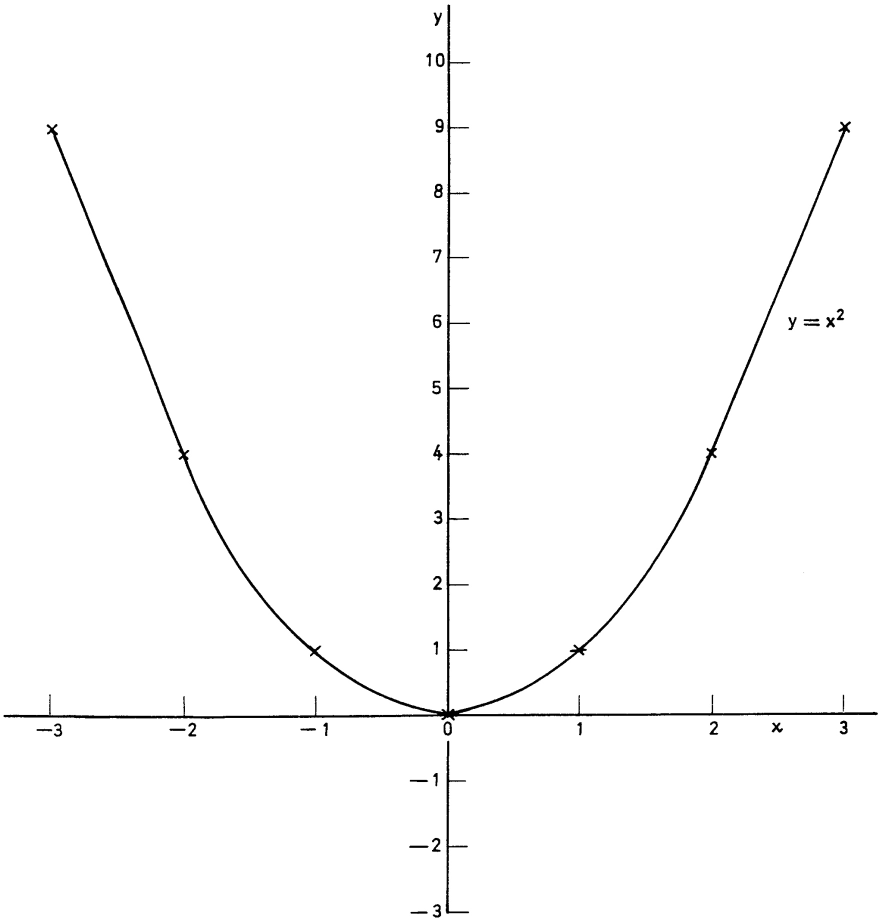
Draw the graph of $y = x^2$ taking whole number values of x from x = –3 to x = 3 inclusive.

As ever, the first step is to draw up a table of values for the given range of x.

x	–3	–2	–1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

Plot these seven points on the graph and then join the points with a smooth curve, as shown in Figure 5.9

Figure 5.9: Graph of $y = x^2$



Example 2

Draw the graph of $y = 2x^2$ from $x = -3$ to 3. Then from the graph find:

(a) the values of x when $y = 10$;

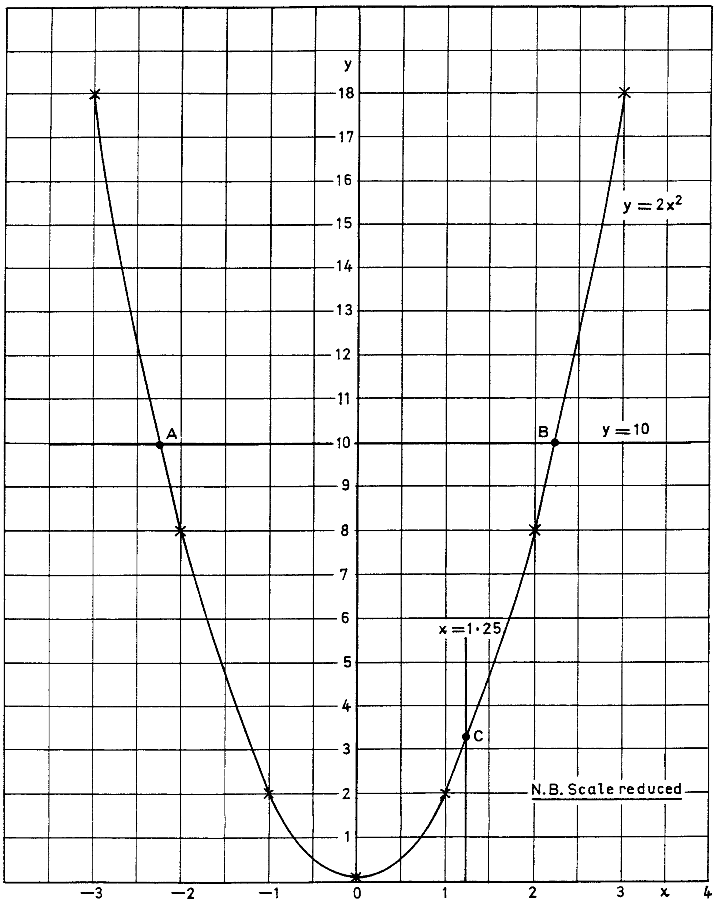
(b) the value of y when $x = 1.25$.

Table of values:

x	−3	−2	−1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = 2x^2$	18	8	2	0	2	8	18

Plot the points and draw the graph as in Figure 5.10.

Figure 5.10: Graph of $y = 2x^2$



Reading off from this:

(a) when y = 10, there are two values of x, at points A and B: x = –2.25 and x = 2.25

(b) when x = 1.25, the value of y is at the point C: y = 3.3

Example 3

Draw the graph of $y = x^2 - 2x + 1$ from x = –2 to 3.

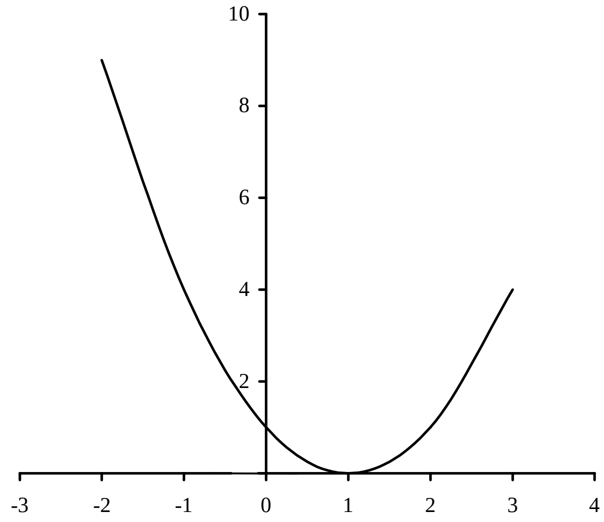
At what value of x does the minimum value of y occur?

Table of values:

x	–2	–1	0	1	2	3
x^2	4	1	0	1	4	9
–2x	4	2	0	–2	–4	–6
+1	1	1	1	1	1	1
$y = x^2 - 2x + 1$	9	4	1	0	1	4

Plot the points and draw the graph as in Figure 5.11.

Figure 5.11: Graph of $y = x^2 - 2x + 1$



Reading from the graph, we can see that the minimum value of y occurs when x = 1.

Practice Questions 3

1. Draw the graph of $y = \frac{1}{2}x^2$ from $x = -4$ to 4.

   From the graph find:

   (a) the value of y when x = 3.2;

   (b) the values of x when y = 7.

2. Draw the graph of $y = -x^2$ from $x = -3$ to 3.

3. Draw the graph of $y = -4x^2$ from $x = -2$ to 2.

   Find the values of x when $y = -9$.

Now check your answers with the ones given at the end of the unit.

Equations with x as the Denominator

These equations take the form of $y = \frac{a}{x}$ where a is a constant. They result in a characteristic curve, the direction of which may vary.

Example 1

Let's look at the case where a = 1, i.e.:

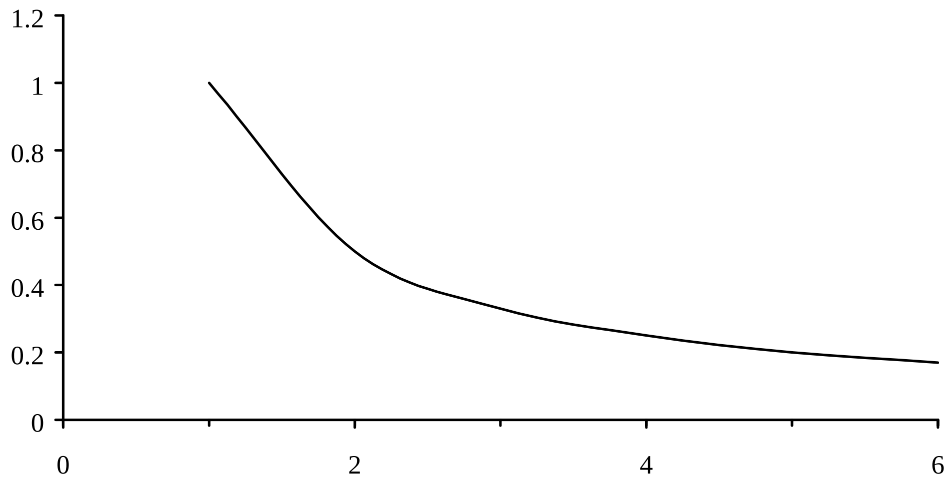
$$y = \frac{1}{x}$$

We'll take x to have values from x = 1 to x = 6, which gives the table below and the graph in Figure 5.12.

x	1	2	3	4	5	6
$y = \frac{1}{x}$	1	0.5	0.33	0.25	0.2	0.17

Note that we have avoided taking x = 0, since $\frac{1}{0}$ is not defined.

Figure 5.12: Graph of $\boldsymbol{y = \frac{1}{x}}$



This graph illustrates the general shape of graphs representing $y = \frac{a}{x}$. In each case, as x gets bigger, y gets smaller. There is also no point corresponding to x = 0.

Example 2

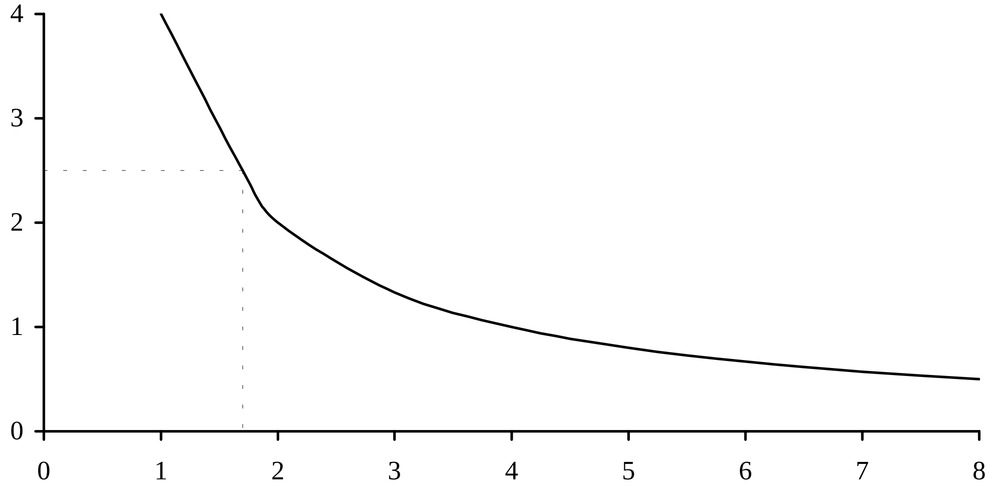
Draw the graph of $y = \frac{4}{x}$ for values of x from x = 1 to x = 8.

From the graph, find the value of x when y = 2.5.

Table of values:

x	1	2	3	4	5	6	7	8
$y = \frac{4}{x}$	4	2	1.33	1	0.8	0.67	0.57	0.5

Figure 5.13: Graph of $\boldsymbol{y = \frac{4}{x}}$



Reading from the graph, when y = 2.5, x = 1.7.

Practice Questions 4

1. Draw the graph of $y = \frac{2}{x}$ from x = 1 to x = 6.

   From your graph, find the value of x when y = 1.5.

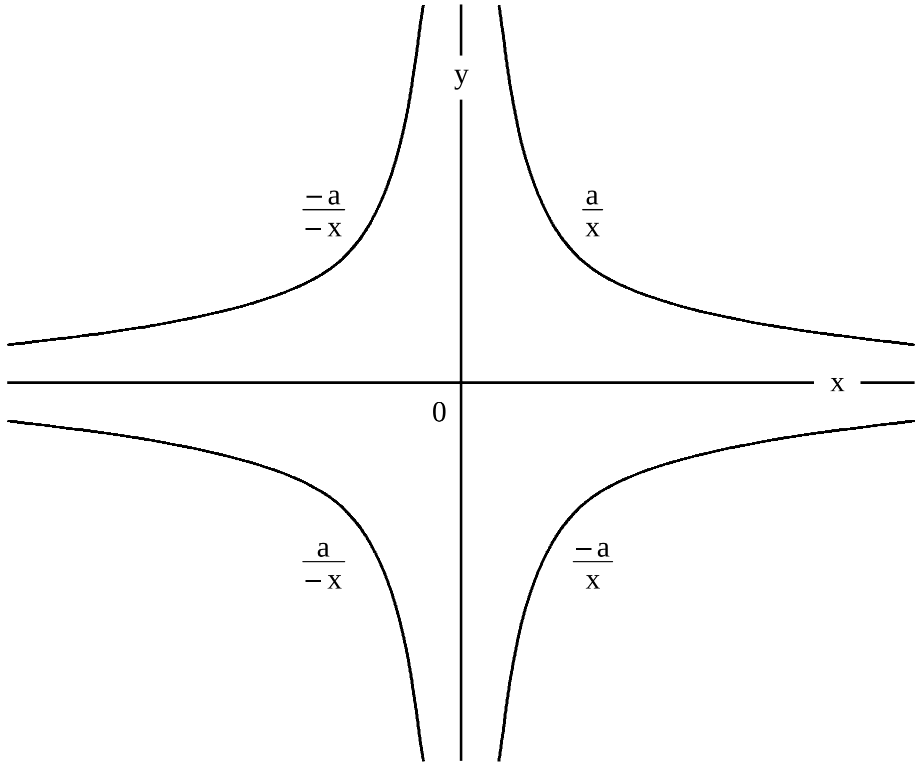
Now check your answers with the ones given at the end of the unit.

Just as we saw that the shape of the quadratic graphs changed according to whether the coefficient of x^2 is positive or negative, so the shape of graphs developed from equations where x is the denominator change:

- $\frac{a}{x}$ (i.e. where both the a and x are positive) gives a graph which slopes down from left to right;
- $\frac{-a}{-x}$ gives a graph which slopes down from left to right;
- $\frac{a}{-x}$ gives a graph which slopes up from right to left;
- $\frac{-a}{x}$ gives a graph which slopes up from right to left.

These are illustrated in Figure 5.14.

Figure 5.14: Shape of graphs from the equation $\boldsymbol{\frac{a}{x}}$



D. GRADIENTS

The gradient of a line is a measure of how much it slopes.

- A horizontal line has a gradient of zero:

- A vertical line has an infinitely large gradient:

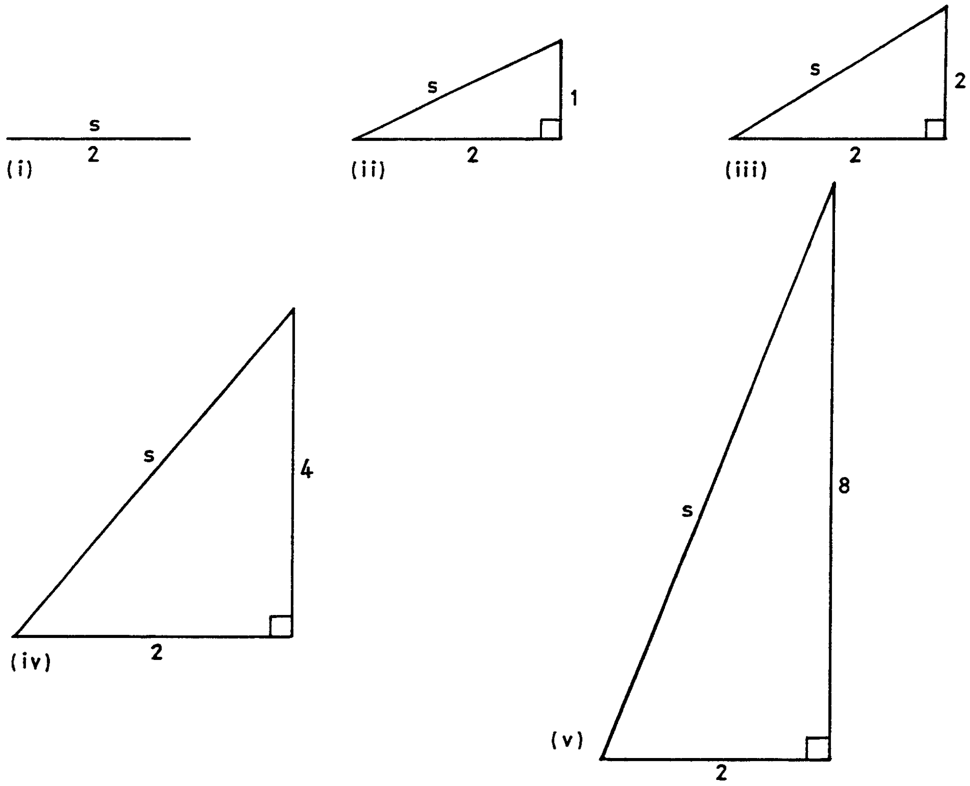
- In between, the gradient increases: 

  or decreases 

Now, consider the five slopes illustrated in Figure 5.15.

In the five shapes, the slope of the line (s) increases as we go from (i) through to (v).

Figure 5.15: Gradient of a slope



In (i), the gradient is zero (no slope).

In the other cases, the gradient is defined as the height of the right-angled triangle divided by the base. Therefore:

In (ii), the gradient of s is $\frac{1}{2} = 0.5$

In (iii), the gradient of s is $\frac{2}{2} = 1$

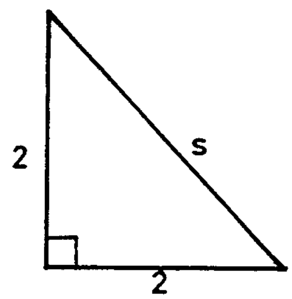
In (iv), the gradient of s is $\frac{4}{2} = 2$

In (v), the gradient of s is $\frac{8}{2} = 4$

If a line slopes ***down*** from left to right (instead of up), then the gradient is ***negative***.

This is denoted by introducing a negative sign, as illustrated by Figure 5.16.

Figure 5.16: Negative gradient



The gradient of the line s is $-\frac{2}{2} = -1$

Gradient of a Straight Line

We can easily calculate the gradient of a straight line joining two points from the co-ordinates of those points.

Firstly, we shall examine the method by reference to plotting the line on a graph.

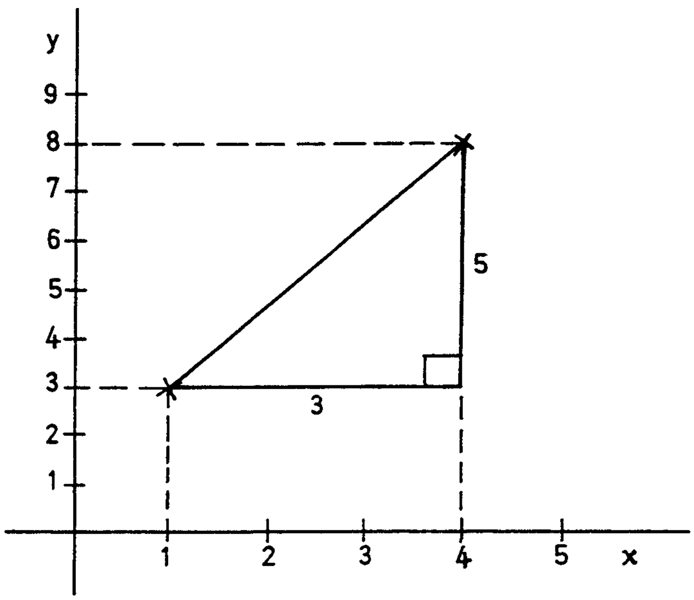
Example 1

Plot the points (1, 3) and (4, 8) and find the gradient of the line joining them.

The procedure would be as follows:

(a) Plot the two points on a graph and join them to give a sloping line, as shown in Figure 5.17.

Figure 5.17: Gradient of the line connecting points (1, 3) and (4, 8)



(b) Make this line the hypotenuse of a right-angled triangle by drawing in a horizontal "base" and a vertical "height".

(c) Work out the lengths of the base and the height from the graph:

Base = 3

Height = 5.

As we have seen, the gradient is given by the equation $\frac{\text{height}}{\text{base}} = \frac{5}{3} = 1\frac{2}{3}$

Note that the slope is positive.

Therefore, the gradient of the straight line joining (1, 3) and 4, 8) is $1\frac{2}{3}$

Note that, at step (c), we could have worked out the lengths of the base and the height from the co-ordinates of the points:

Length of the base = Difference between x co-ordinates = 4 – 1 = 3

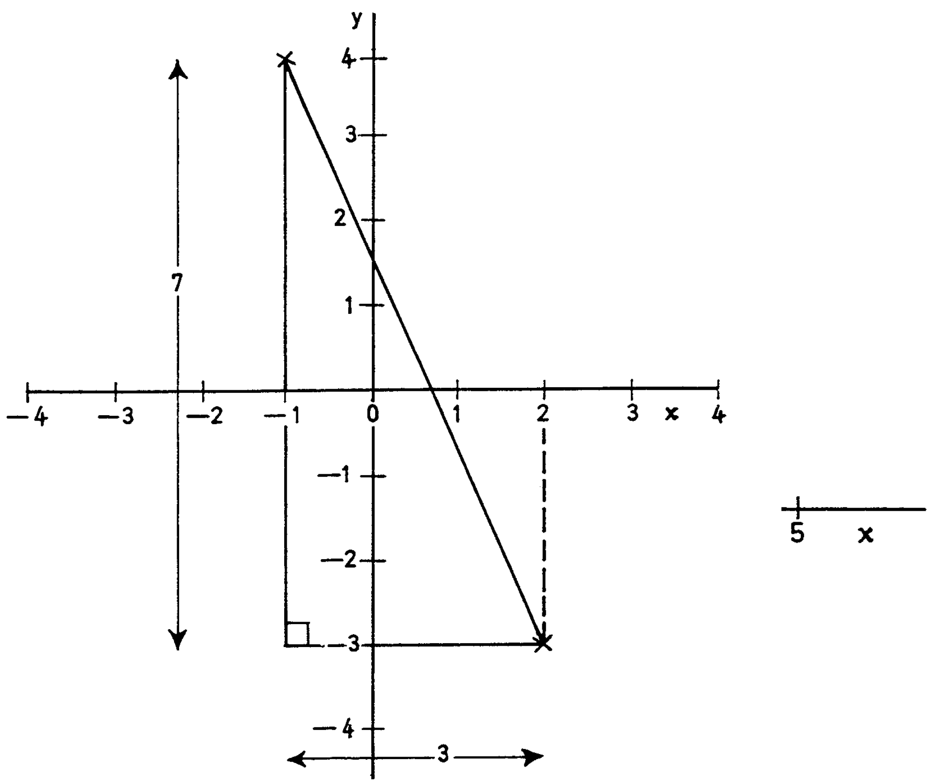
Length of the height = Difference between y co-ordinates = 8 – 3 = 5

Example 2

Plot the points (–1, 4) and (2, –3) and find the gradient of the line joining them.

Plot the points, join them and construct a right-angled triangle with the line forming the hypotenuse:

Figure 5.18: Gradient of the line connecting points (–1, 4) and (2, –3)



This time we shall work out the lengths of the base and the height from the co-ordinates of the points.

Base = 2 – (–1) = 3

Height = 4 – (–3) = 7

We can check that this is correct by reference to the plotted triangle on the graph.
Note that the slope is negative.

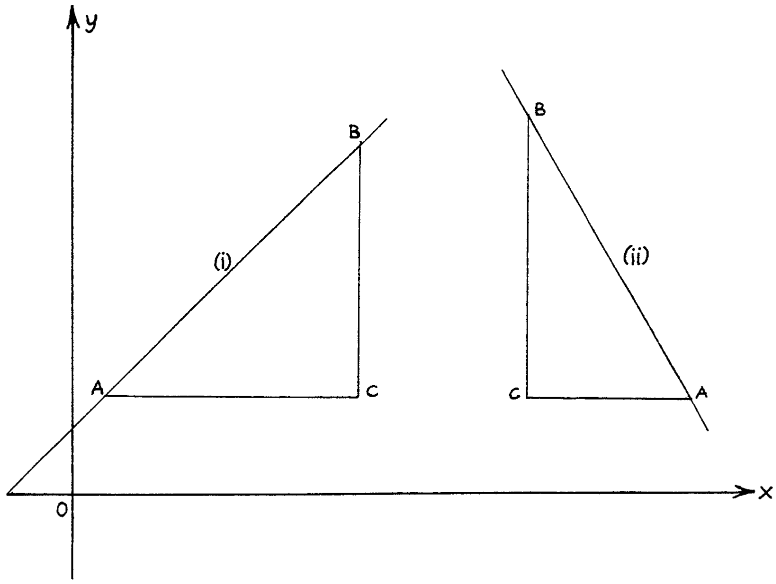
Therefore, the gradient $= -\dfrac{\text{height}}{\text{base}} = -\dfrac{7}{3} = -2\frac{1}{3}$

General formula

We can now develop a general formula for the gradient of a straight line by reference to its co-ordinates.

To find the gradient of a line, take any two points A and B on the line. Draw AC parallel to the x axis, and CB parallel to the y axis. The gradient is then given by the ratio CB : AC.

Figure 5.19: Gradient of a straight line



Note that these lengths are ***directed*** lengths – i.e. AC is positive if the movement from A to C is in the same direction as O to x, and negative if the movement from A to C is in the opposite direction to O to x. In the same way, CB is positive if the movement from point C to B is in the same direction as O to y.

Taking the lines in Figure 5.19, we find the following:

- In line (i), AC and CB are both positive, hence the gradient $\dfrac{CB}{AC}$ is positive.
- In line (ii), AC is negative, CB is positive, and the gradient is negative.

We can further seen that AC is the difference between the co-ordinates along the x axis and CB is the difference between the co-ordinates along the y axis. Therefore, we if the points are (5,1) and (7,2) the gradient is:

$$\frac{2-1}{7-5} = \frac{1}{2}.$$

For points (–1, –2) and (2, –3), the gradient is:

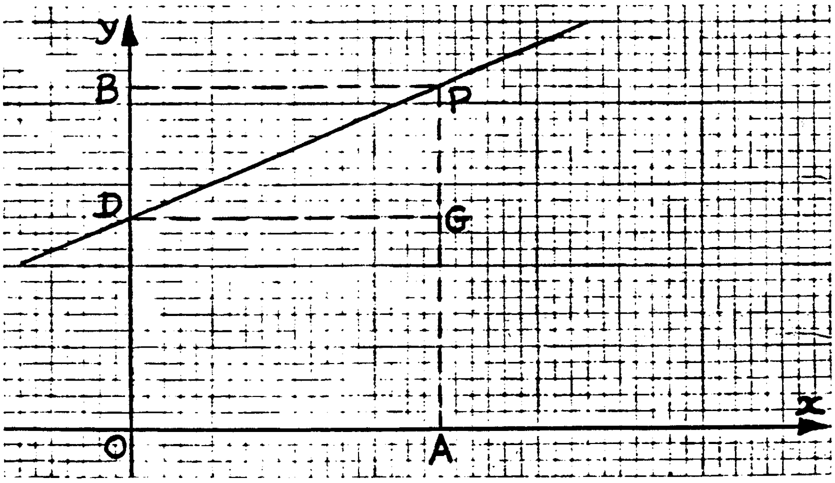
$$\frac{-3-(-2)}{2-(-1)} = \frac{-3+2}{2+1} = -\frac{1}{3}.$$

Equation of a Straight Line and Gradient

We can now go back to consider aspects of the equation of a straight line which we have met before.

Consider a general straight line drawn on a graph. D is the point on that line where it crosses the y axis, and P is another point along the line. This is shown in Figure 5.20.

Figure 5.20: Proving the equation of a straight line



OAPB is a rectangle and OAGD is a rectangle.

Let: OA = x, OB = y, the intercept OD = c and the gradient of DP = m.

Then:

$$m = \frac{PG}{DG} = \frac{PG}{OA} = \frac{BD}{OA}$$

$$= \frac{BO - DO}{OA} = \frac{y - c}{x}$$

Rearranging the equation $m = \frac{y-c}{x}$ we get:

$$mx = y - c$$

$$y = mx + c$$

This is the general equation of any straight line. Note that c is the intercept that the line makes with the y axis. If the line cuts the y axis below the origin, its intercept is negative, –c.

Using this method, given any straight line graph, its equation can be found immediately.

Also, given any linear equation, we can find the gradient of the line (m) immediately. For example:

Find the gradient of a line with the equation 6x + 3y = 19.

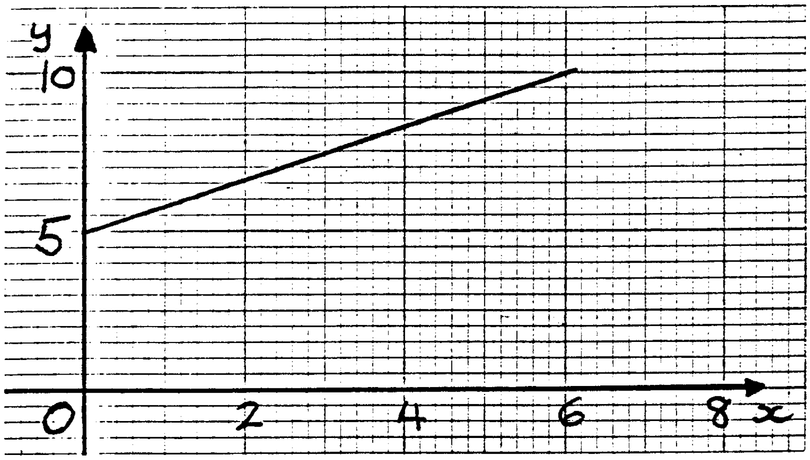
We need to put the equation into the general form y = mx + c:

$$y = \frac{-6x + 19}{3}$$

Therefore, the gradient is $\frac{-6}{3} = -2$

Finally, consider the graph in Figure 5.21.

Figure 5.21: Deriving the equation of a straight line



The gradient of the line, m $= \frac{10-5}{6} = \frac{5}{6}$

The intercept, c = 5.

Therefore, the equation of this line is:

$$y = \frac{5}{6}x + 5$$

or $6y = 5x + 30$

Practice Questions 5

1. Find the gradients of the straight lines joining the following points.

   (a) (1, 3) and (3, 7)

   (b) (2, −1) and (5, 6)

   (c) (−2, 6) and (1, −1)

   (d) (−4, 5) and (2, 0)

   (e) (7, 0) and (−1, −4)

   (f) (0, −2) and (4, 7)

2. Find the gradients of the lines with the following equations:

   (a) $2y - 3x + 6 = 0$

(b) $7y = 4x$

(c) $7y + 1 = 2x$

The next three questions bring together aspects of the whole unit so far. They are actual examination questions.

3. Draw an x-axis from −3 to 5 and a y-axis from 4 to −4. On them draw the graphs of $y = x - 1$ and $2y = 4 - x$. Write down the co-ordinates of the point where the two lines intersect.

4. Draw the graph of $y = \frac{3x^2}{2}$ for values of x from −3 to +3.

   On the same axes draw the line $y = 5$. Write down the co-ordinates of the points where the line and the curve intersect.

5. Plot the points (6, 5) and (3, 3). Find (a) the gradient and (b) the equation of the straight line joining them.

Now check your answers with the ones given at the end of the unit.

F. USING GRAPHS IN BUSINESS PROBLEMS

In this final part of the unit, we shall consider how we can use the properties of graphs we have been examining to analyse and solve business problems

Business Costs

All businesses incur costs in the production and supply of the goods and/or services which they provide. These are an important consideration in many aspects of business finance, not least in calculating profit which, at heart, is simply:

Revenue from sales *less* Costs of production and supply (or Cost of sales)

The analysis of costs is a subject of its own and, within that, there are many ways of looking at costs. One of these is to divide them into ***fixed*** and ***variable*** costs.

- Fixed costs are those which the business must always meet, regardless of the number of units produced – such as rent on buildings.
- Variable costs are those which vary with output – such as the cost of raw materials used as part of the production process.

Thus, if we take the case of a printing company, its fixed costs would include the rent on its offices, printing room and warehouse, and the variable costs would include paper and ink. Whether they print six or six hundred books a week, the fixed costs remain the same, but the variable costs rise with the number of books produced. (Note that fixed costs are only fixed in the short term – over a long period, they may change by, say, acquiring different premises which may be cheaper.)

If we consider variable costs, we can say that they are a function of the level of production. This means that, as the level of production changes, so do the variable costs.

We can turn this observation into an algebraic expression:

let: x = Number of units produced (level of production)

b = Variable costs

Then:

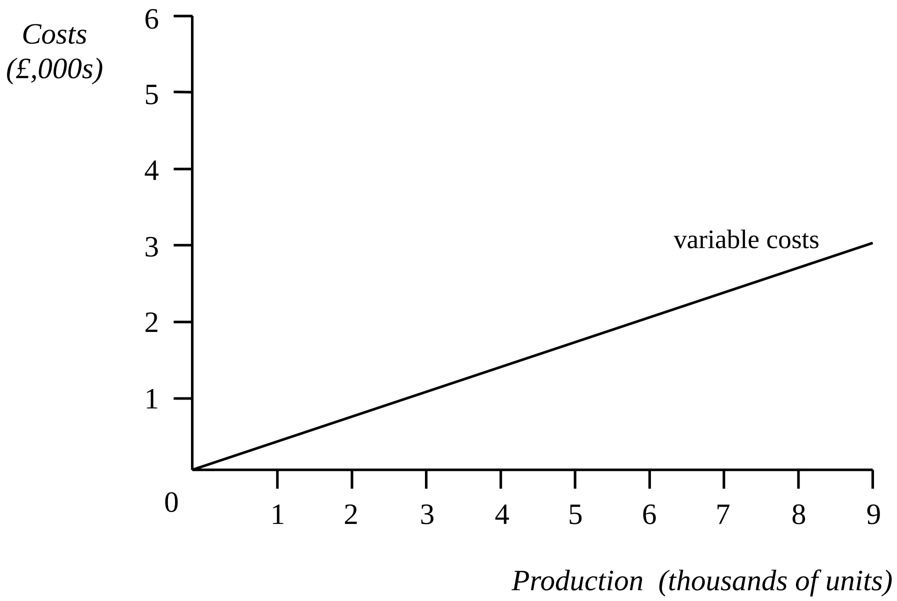
b = f(x)

This is where we started the unit!

We know that we can draw a graph of this relationship if we know some of the values for x and y.

Let us suppose that we do have such values and can produce a such a graph (Figure 5.22):

Figure 5.22: Graph of relationship between variable costs and level of production



We can work out the gradient of the line on this graph, based on two co-ordinates. One of those co-ordinates must be (0, 0) since when there is no production, no variables costs will be incurred. If we known that one other point on the graph is (9, 3), then the gradient (which is m) is:

$$m = \frac{3-0}{9-0} = \frac{3}{9} = \frac{1}{3}$$

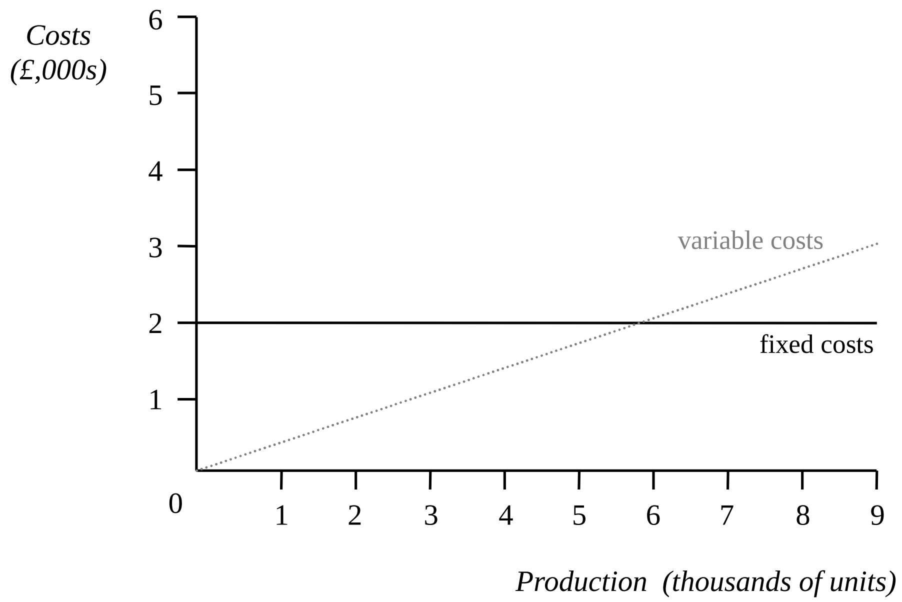
We also know that the intercept (c) of the line is 0, so we can express the relationship of variable costs to units of production by the following equation in the form of y = mx + c:

$$a = \tfrac{1}{3}x + 0$$

$$\text{or } 3a = x$$

Now consider fixed costs. These do not vary with the level of production. So, if we know the level of fixed costs, we can show this on the same graph as above:

Figure 5.23: Graph of fixed costs and level of production



We can see that this line is parallel to the x axis, with an intercept at 2. This is represented by the equation:

$$b = 2 \text{ (where b is fixed costs)}$$

Businesses are not only interested in costs as either fixed or variable, but in their total costs. We can say that:

Total costs = Variable costs + fixed costs

If total costs = y, and using the notation from above, we get the following equation:

$$y = a + b$$

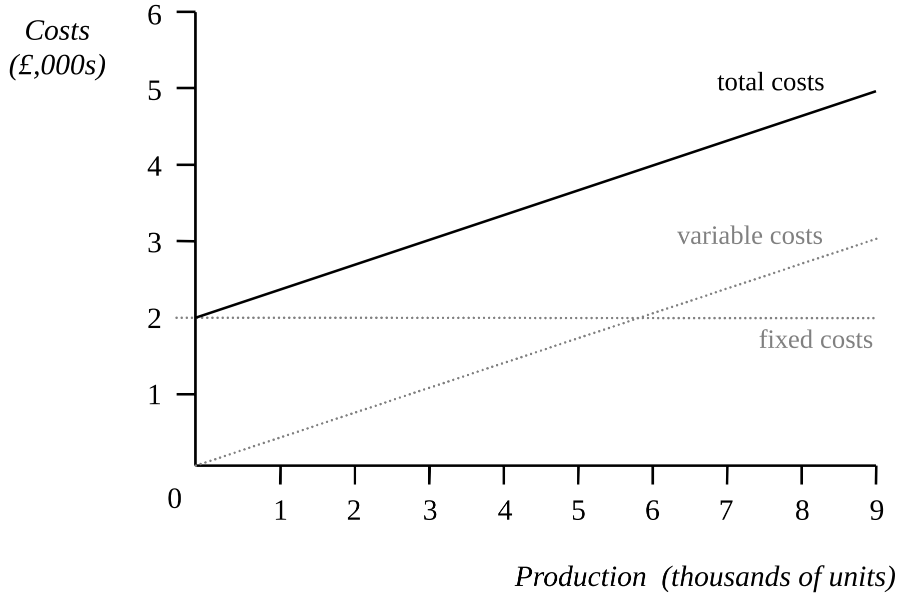
Substituting from the equations established for a and b, we get:

$$y = \frac{1}{3}x + 2$$

This is a standard linear equation. We could work out a table of values for it (which we shall skip here) and draw a graph. This has been done, again on the same graph as above, in Figure 5.24.

Note that the line for total costs is, in fact, a combination of the lines for variable and fixed costs. Its intercept is at the level of fixed costs and its gradient is that of variable costs.

Figure 5.24: Graph of total costs and level of production



Note that the chart is only a model, which is why we can show the variable (and total) costs lines as linear. In real life, variable costs do not necessarily rise at a constant rate with increases in production. Despite this, though, it is a useful way of looking at costs.

Consider the following situation.

A business wishes to use a courier company on a regular basis. Company A charge a call out charge of £5.00 plus £1.00 per mile, whilst company B charge £2.00 per mile but no call out charge. Which one is more economical?

We can develop equations for these charges and then use a graph to show which company would be more economical at different journey lengths:

Let: $x = $ Number of miles

$y = $ Total cost (£)

Then, for company A, total cost may be defined as :

$$y = x + 5$$

For company B, the costs is:

$$y = 2x$$

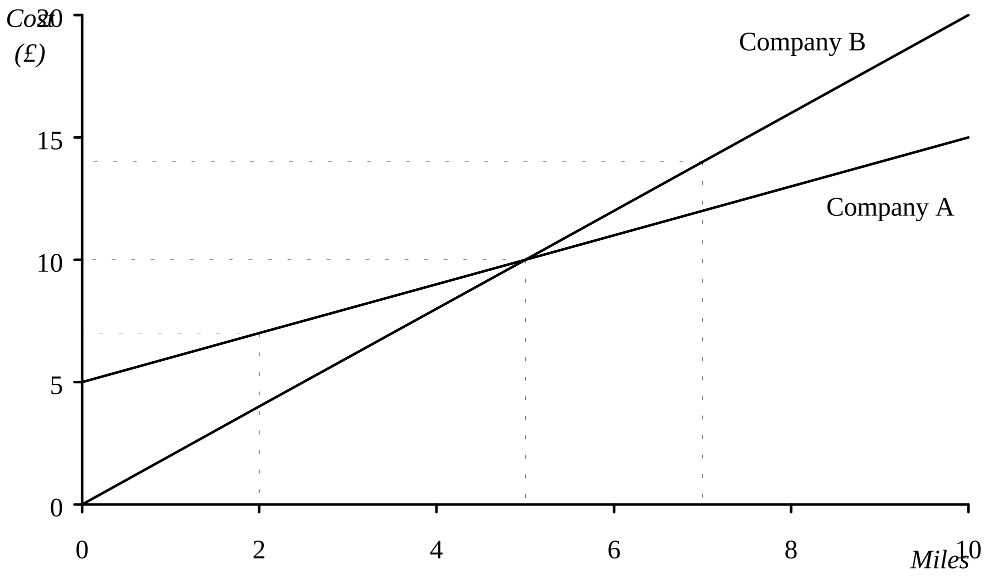
The graph is shown in Figure 5.25.

The point at which the two lines intersect means that the values of x and y are the same for each line. Thus, at that point, the cost of using each company is the same. This takes place at 5 miles. For any distance above this, company A is cheaper, whereas for distances less than five miles, it is cheaper to use company B.. For example:

- at 12 miles, A charges £12 and B charges £14;
- at 2 miles, A charges £7 and B charges £4.

These are shown by dotted lines on the graph.

Figure 5.25: Comparison of courier company charges



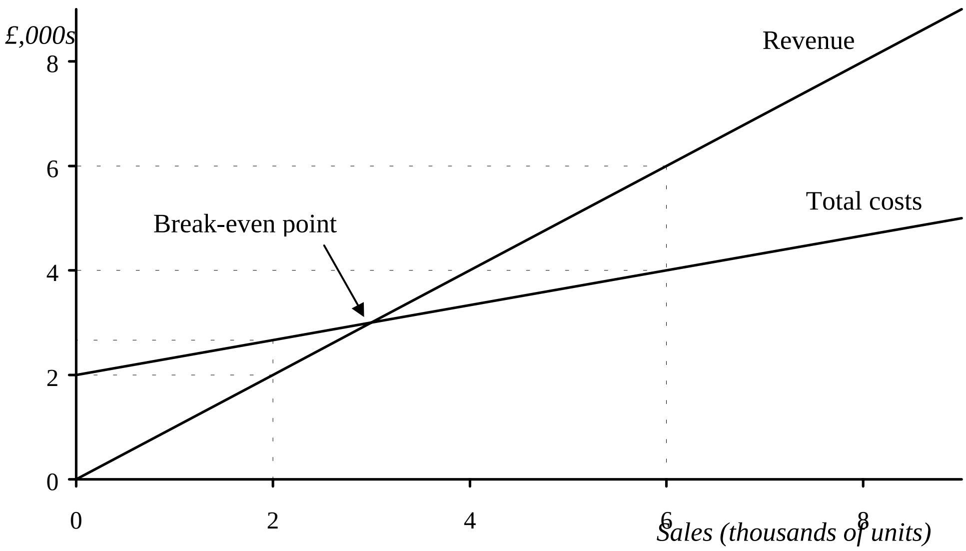
Break-even Analysis

We can further develop the costs graph by including revenue on it.

Revenue is the total income from sales of the units produced. If we assume that each unit sells for £1, we can say that y = x where y is the income from sales (in £) and x is the number of units sold.

Adding the graph of this to the costs graph shown in Figure 5.25, we get Figure 5.26. (Note that we are assuming that all units produced are sold.)

Figure 5.26: Costs and revenue graph



The point at which the revenue and cost lines intersect is known as the ***break-even point***. At that point, revenue exactly equals costs. Here, this takes place at sales of 3,000 units, with both costs and revenue being £3,000.

Any sales above the break-even point will mean that revenue exceeds costs, and there will be a profit. At sales levels below the break-even point, though, costs exceed revenue and there will be a loss. For example:

- at sales of 2,000 units, costs are £2,667 whilst revenue is only £2,000 – there will a loss of £667;
- at sales of 6,000 units, costs are £4,000 whilst revenue is £6,000 – there will be a profit of £2,000.

The break-even point represents the minimum position for the long term survival of a business.

Practice Questions 6

1. Teapots R U Ltd. makes and sells a small range of teapots. It has worked out that the following costs for its best selling teapot "Standard" are:

   Fixed costs = £20,150

   Variable costs = £5.50 per "Standard" teapot.

   If the price of a Standard Teapot is £12.00, how many must Teapots R U Ltd. sell each week to break even? Use a fixed and variable costs chart (break-even chart) to find the answer.

2. A company makes bicycles and sells them for £80.00. The fixed cost of producing them is £80,000 with a variable cost per bicycle of £36.00. Using a fixed and variable costs chart (break-even chart), find the number of bicycles needed to be sold for the company to break even. Give your answer to the nearest 100 bicycles.

Now check your answers with the ones given at the end of the unit.

Economic Ordering Quantities

In this last section of the unit we shall look at the role of graphs in analysing and helping to solve a purchasing problem. We shall not work through the problem in detail, but just review the general approach and the way in which graphs can be used.

One of the central issues in purchasing materials to be used as part of the production process, is how much to order and when. Many producers and most retailers must purchase the materials they need for their business in larger quantities than those needed immediately. However, there are costs in the ordering of materials – for example, each order and delivery might would incur large administrative and handling costs, including transportation costs, so buying-in stock on a daily basis might not be appropriate. On the other hand, there are also costs involved in holding a stock of unused materials – very large stocks can incur large warehousing costs. Set against this, buying in larger quantities has the effect of spreading the fixed costs of purchase over many items, and can also mean getting cheaper prices for buying in bulk. The purchaser must determine the optimum point.

By "economic order quantity" we mean the quantity of materials that should be ordered in order to minimise the total cost. In order to do this, we need to look at the relationships between the usage, supply and costs involved.

Suppose we have the following problem:

- We want to order 3,600 units over one year and these units are used at a constant rate.

- As soon as we order, we receive the whole order instantaneously – unrealistic but let us keep the problem simple.
- We order as soon as existing stock runs out.
- The cost of placing the order is £30.
- The cost of keeping one unit in stock for a year is £2.

We have several options as to how we could tackle this:

(a) Place a single order for the whole year.

(b) Make 12 monthly orders.

(c) Make three orders at four-monthly intervals.

(d) Make six orders bi-monthly.

(e) Make 36 orders at 10-day intervals.

There are many other possibilities but these will serve our purpose.

The relevant costs for each option above are:

(a) £30 for one order + storage costs of £2 × 1,800 = £3,630

(As we place an order for 3,600 units and use them evenly through the year, the average stock held in storage will be 1,800 units.]

(b) (12 × £30) for 12 orders + (£2 × 150) storage = £660

(In this case, the orders are for 300 each, so the average holding is 150.)

(c) (3 × £30) + (£2 × 600) = £1,290

(600 is the average holding on orders of 1,200 each.)

(d) (6 × £30) + (£2 × 300) = £780

(Six orders of 600 each, giving the average holding of 300.)

(e) (36 × £30) + (£2 × 50) = £1,180

(36 orders of 100 each, giving the average holding of 50.)

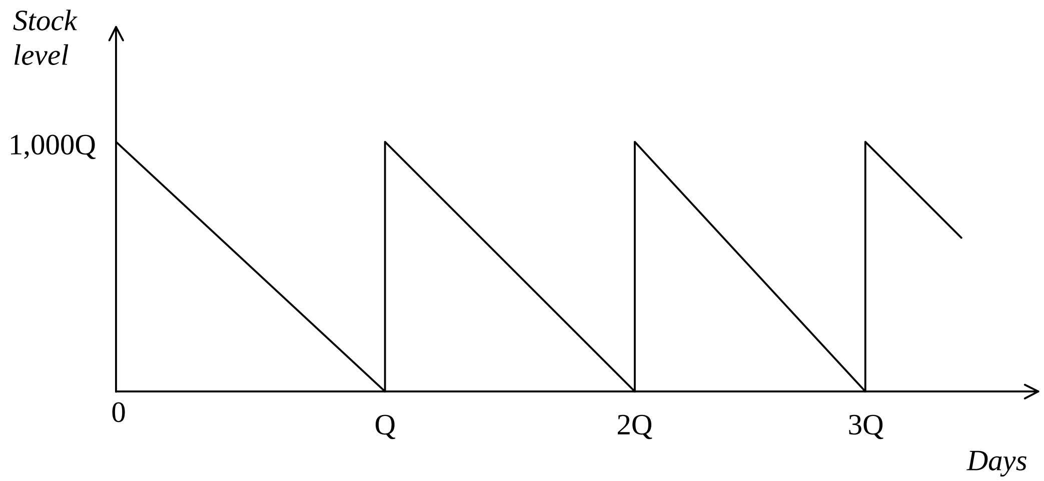
We can see that the cheapest option is (b), i.e. monthly orders of 300. But is it really the cheapest? We would need to try many other options before we could be sure.

Let us consider another problem.

A company called Oily Engineers Ltd uses 1,000 sprockets every day. It could purchase daily, but considers this to be unacceptable as the cost of raising an order and sending a van to a cash-and-carry to collect the goods has been estimated at £100.

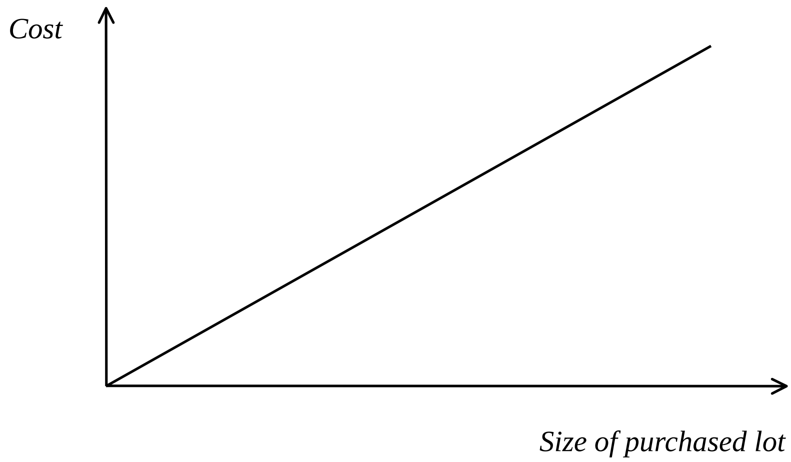
If it buys in lots of 1,000Q sprockets, then the stock will last for Q days. The graph of stock level is a saw-tooth. There will be 1,000Q sprockets in stock immediately after a purchase has been made. Q days later this will be down to zero, when another purchase will be made. The average stock level is 500Q. (See Figure 5.27.)

Figure 5.27: Stock levels over time



The warehousing cost is linear – if the amount purchased in each order is doubled, then the warehousing costs are doubled as there are, on average, twice as many items in the warehouse. This shown in Figure 5.28.

Figure 5.28: Warehousing costs



The cost of purchasing is not linear. The cost per annum depends on the number of orders placed, as it is a fixed amount per order.

For example, assume that there are 250 working days each year:

(a) Sprockets purchased in batches of 1,000, i.e. a batch every day:

Cost = £100 × 250 = £25,000

(b) Sprockets purchased in batches of 2,000, i.e. a batch every other day:

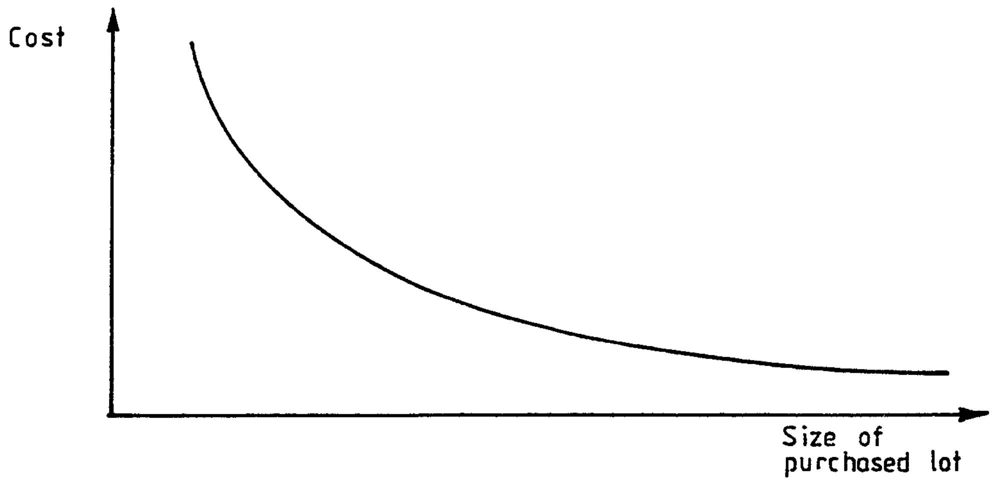
Cost = £100 × 125 = £12,500

(c) Sprockets purchased in batches of 4,000, i.e. a batch every fourth day:

Cost = £100 × 62.5 = £6,250

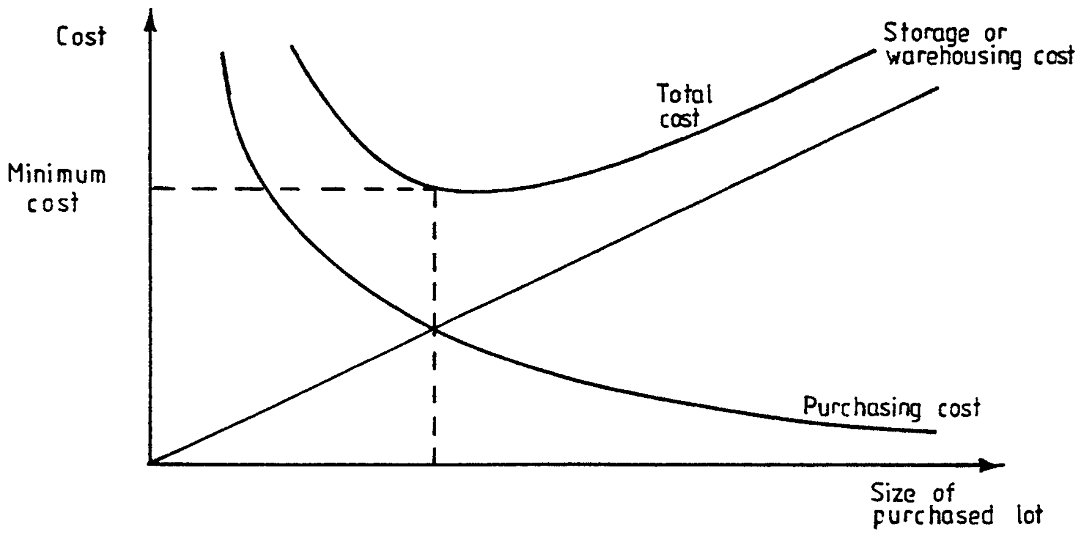
Each time the amount purchased is doubled, the cost per annum halves. The graph is shown in Figure 5.29.

Figure 5.29: Cost of Purchasing (per Annum)



These costs can be added to the warehouse costs to give a total cost per annum. The graph is this is shown in Figure 5. 30.

Figure 5.30: Total costs of purchasing



Thus, we can identify the economic order quantity from the minimum cost position on the graph.

ANSWERS TO QUESTIONS FOR PRACTICE

Practice Questions 1

1.

x	0	0.5	1	1.5	2	2.5
y = 3x – 2	–2	–0.5	1	2.5	4	5.5

2. y = 2x + 1 is the only function which satisfies **all** values in the table.

3. (a) f(2) = 10 – 2 = 8

   (b) f(–3) = –15 – 2 = –17

4. (a) f(2) = 4 + 2 = 6

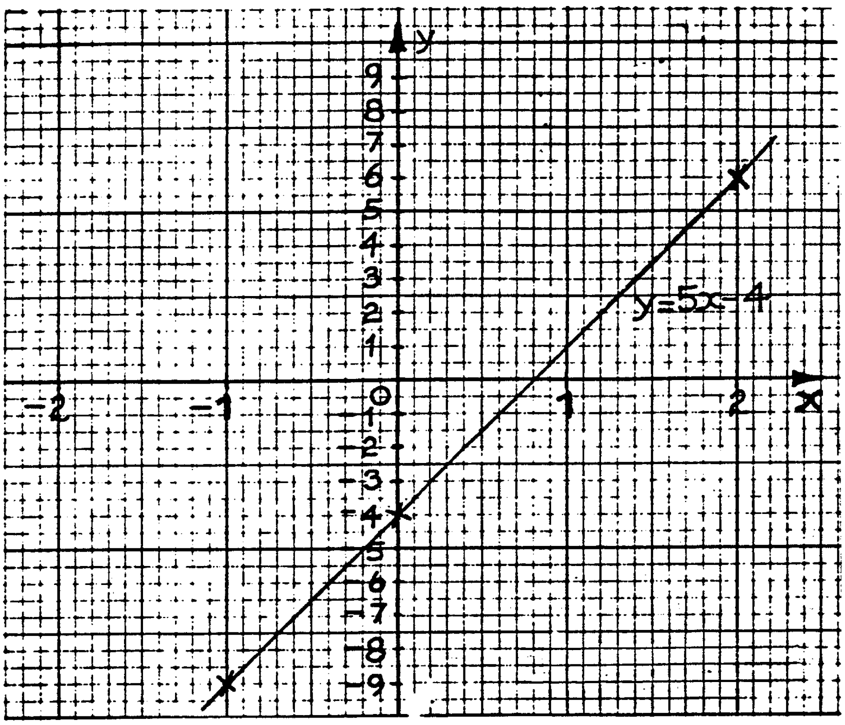
   (b) f(–3) = 9 + 2 = 11

Practice Questions 2

1. Table of values:

x	–1	0	2
y = 5x – 4	–9	–4	6

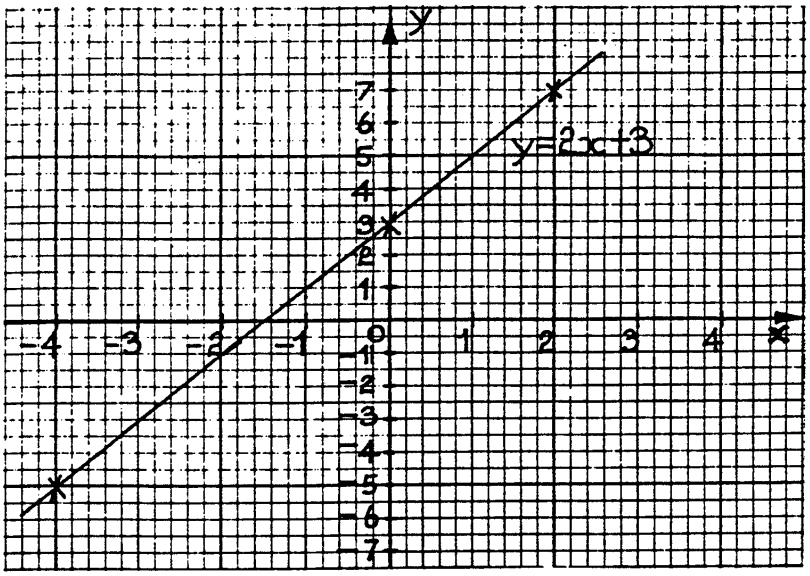
Graph of y = 5x – 4



2. Table of values:

x	–4	0	2
y = 2x + 3	–5	3	7

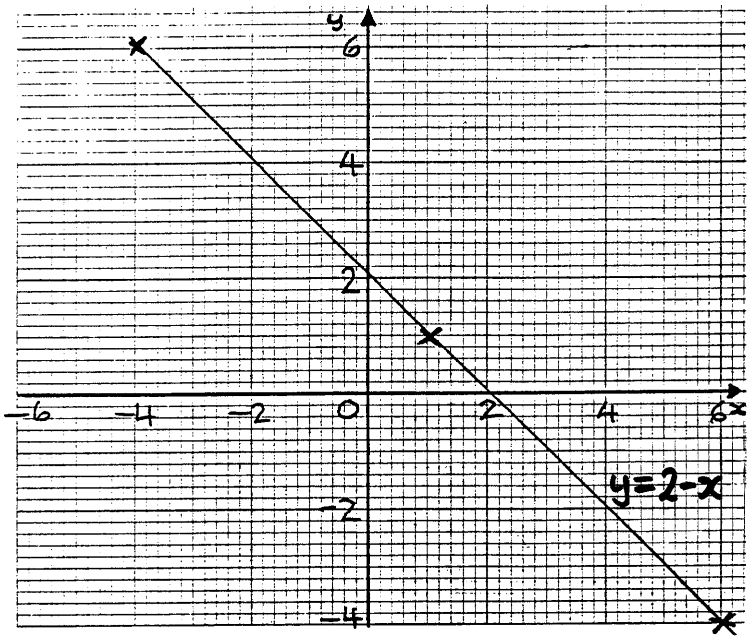
Graph of y = 2x + 3



3. Table of values:

x	–4	1	6
y = 2 – x	6	1	–4

Graph of y = 2 – x



Note that for the next four questions, we do not show the graphs themselves.

4. (i) cuts (iii) at $(1\frac{2}{3},\ 2\frac{2}{3})$ or (1.7, 2.7) to 1 decimal place.

   (i) cuts (iv) at (2.6, 0.8).

   (ii) cuts (iii) at (–1, 0).

   (ii) cuts (iv) at (–1.4, 2.1) to 1 decimal place.

5. (i) y = 5 is a line parallel to the x axis and 5 units above it.

   (ii) x = –5 is a line parallel to the y axis and 5 units to the left of it.

   (iii) y = 5x is a line through the origin, sloping up from left to right with a gradient 5 in 1 (i.e. if you go along 1 unit you go up 5 units).

   (iv) $y = -\frac{1}{5}x$ is a line through the origin, sloping down from left to right with a gradient of 1 in 5 (i.e. along 5 and down 1).

6. These three graphs are parallel lines:

   (i) passes through the origin.

   (ii) makes an intercept of 3 on the y axis and –1½ on the x axis (i.e. crosses the y axis at 3 and the x axis at –1½).

   (iii) makes an intercept of –4 on the y-axis and 2 on the x-axis.

7. The lines intersect at (3, 1).

Practice Questions 3

1. Table of values:

x	–4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
$y = \frac{1}{2}x^2$	8	4½	2	½	0	½	2	4½	8

The graph is shown on the next page. Reading off from it, we get:

(a) When x = 3.2, y = 5.1.

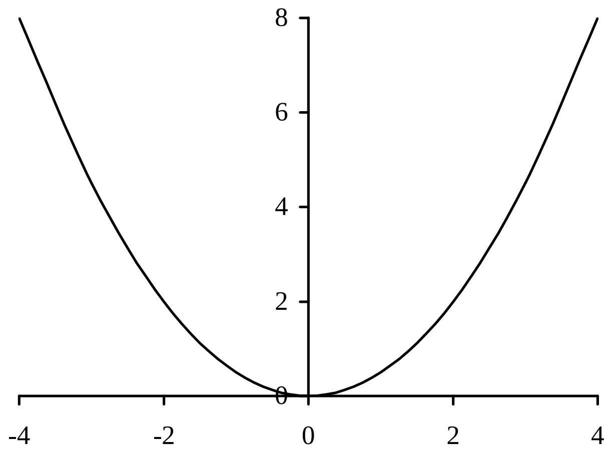
(b) When y = 7, x = –3.7 and 3.7

2. Table of values:

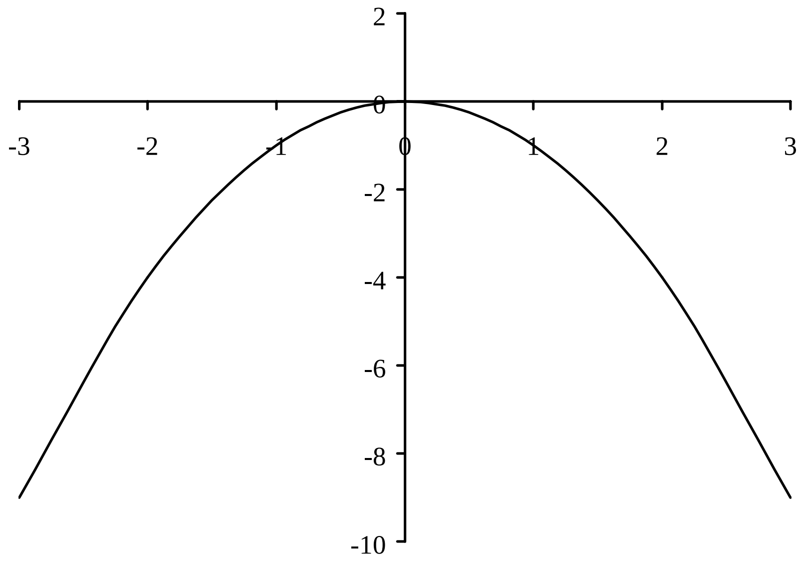
x	–3	–2	–1	0	1	2	3
$y = -x^2$	–9	–4	–1	0	–1	–4	–9

The graph is shown on the next page.

Graph of $y = \frac{1}{2}x^2$



Graph of $y = -x^2$

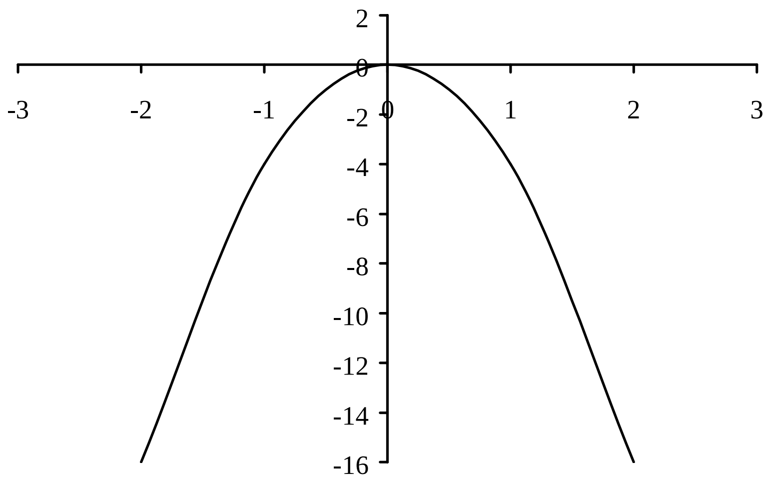


3. Table of values:

x	–2	–1	0	1	2
x^2 $4x^2$	4 16	1 4	0 0	1 4	4 16
$y = -4x^2$	–16	–4	0	–4	–16

The graph is shown on the next page.

Graph of y = –4x²



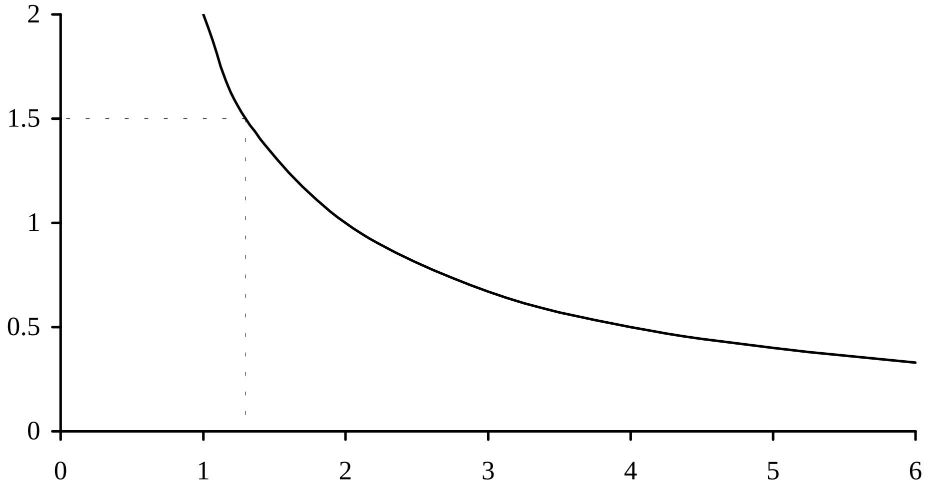
Reading from the graph, when y = –9, x = = –1.5 and 1.5.

Practice Questions 4

1. Table of values:

x	1	2	3	4	5	6
$y = \frac{2}{x}$	2	1	0.67	0.5	0.4	0.33

Graph of $y = \frac{2}{x}$



When y = 1.5, x = 1.3.

Practice Questions 5

1. (a) $\dfrac{7-3}{3-1} = \dfrac{4}{2} = 2$

   (b) $\dfrac{6-(-1)}{5-3} = \dfrac{7}{3} = 2\tfrac{1}{3}$

   (c) $\dfrac{6-(-1)}{(-2)-1} = \dfrac{7}{-3} = -2\tfrac{1}{3}$

   (d) $\dfrac{5-0}{(-4)-2} = \dfrac{5}{-6} = -\dfrac{5}{6}$

   (e) $\dfrac{0-(-4)}{7-(-1)} = \dfrac{4}{8} = \dfrac{1}{2}$

   (f) $\dfrac{7-(-2)}{4-0} = \dfrac{9}{4} = 2\tfrac{1}{2}$

2. (a) $y = \dfrac{3x-6}{2}$ $\quad\therefore$ Gradient $= \dfrac{3}{2}$

   (b) $y = \dfrac{4}{7}x$ $\quad\therefore$ Gradient $= \dfrac{4}{7}$

   (c) $y = \dfrac{2x-1}{7}$ $\quad\therefore$ Gradient $= \dfrac{2}{7}$

3. Tables of values:

x	–3	1	5
y = x – 1	–4	0	4

$2y = 4 - x$

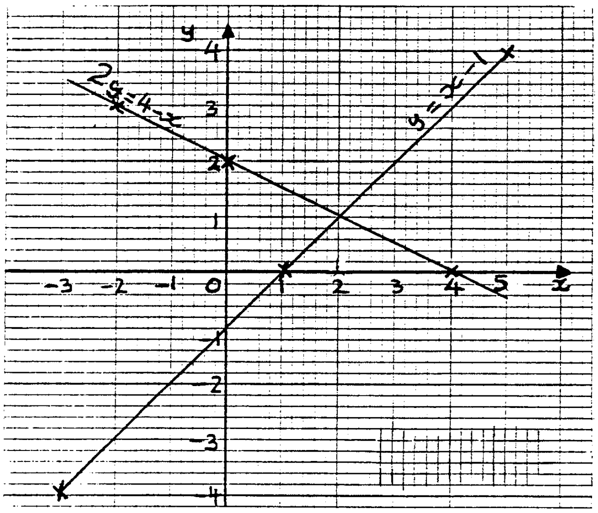
$y = 2 - \tfrac{1}{2}x$

x	–2	0	4
2 $\tfrac{1}{2}$ x	2 –1	2 0	2 2
y = 2 – $\tfrac{1}{2}$ x	3	2	0

The graph is shown on the next page.

Reading from this, the lines intersect at point (2, 1)

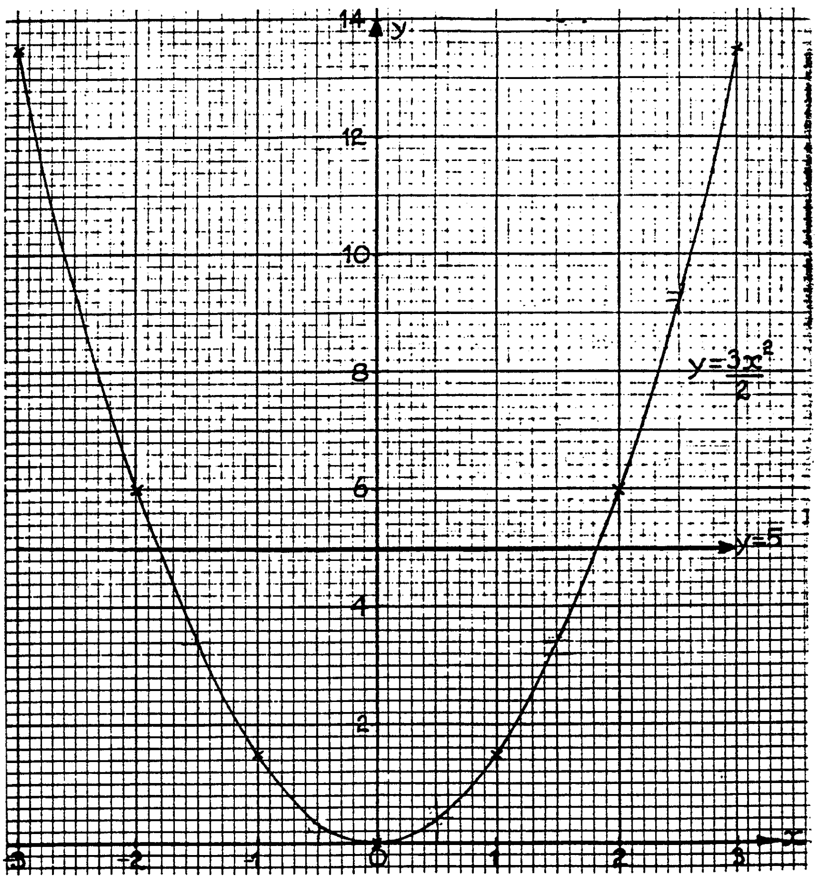
Graph of y = x − 1 and 2y = 4 − x



4. Table of values:

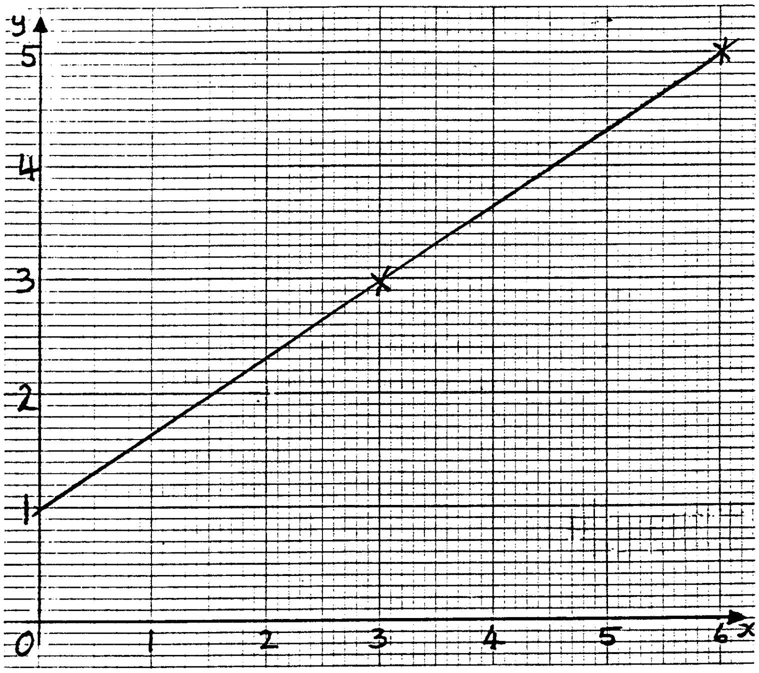
x	–3	–2	–1	0	1	2	3
$y = \frac{3x^2}{2}$	13.5	6	1.5	0	1.5	6	13.5

Graph of $y = \frac{3x^2}{2}$ ***and y = 5***



From the graph, you will see that the line and curve intersect at (1.8, 5) and (–1.8, 5).

5.



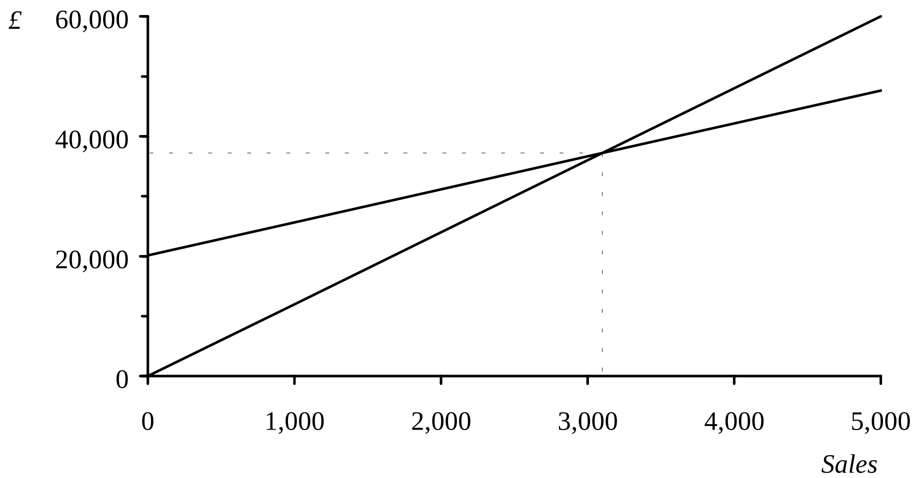
(a) Gradient $= \frac{5-3}{6-3} = \frac{2}{3}$

(b) The line cuts the y-axis at (0, 1).

Therefore, the equation is: $y = \frac{2}{3}x + 1$ *or* $3y = 2x + 3$

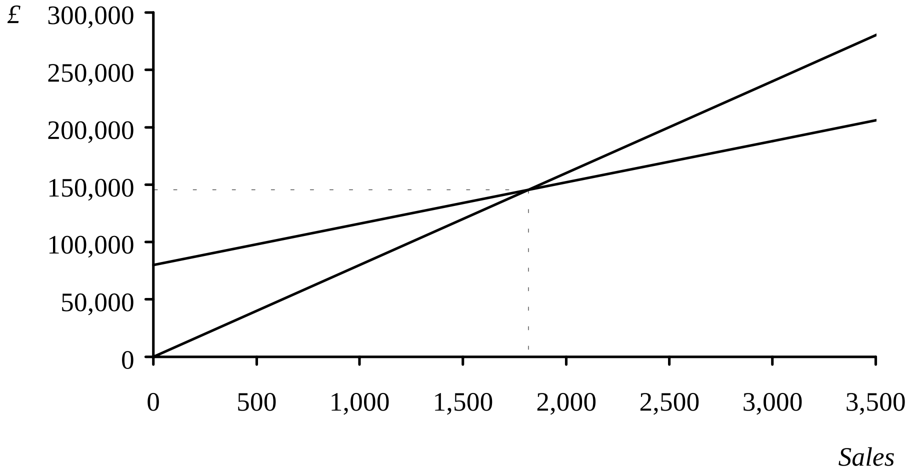
Practice Questions 6

1. Break-even chart:



The break-even point is at sales of 3,100 Standard teapots.

2. Break-even chart



The break-even point is at sales of 1,800 bicycles (to the nearest 100).

Study Unit 6

Introduction to Statistics and Data Analysis

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INTRODUCTION

This unit is the first of five concerned specifically with statistics.

"Statistics" is a word which is used in a variety of ways and with a variety of meanings, but, in whatever way it is used, it is always concerned with numerical information. There are two particular meanings of the word which concern us, namely:

- The numerical facts themselves – for example, we talk of the "statistics" of steel production.
- The methods of analysing the facts – in this sense, "statistics" is the title of a subject like "arithmetic" or "chemistry" or "physics". Sometimes the subject is called "statistical method".

Thus, statistics is concerned with numerical facts about a problem or issue – the facts themselves and methods of analysing them in order to solve or shed light on the problem or issue.

The numerical facts with which statistics is concerned are commonly called ***data***. So, therefore, we talk about the collection and analysis of data in respect of a particular problem.

The methods used for analysing data can be divided into two categories – descriptive and inferential.

- ***Descriptive methods*** simply manipulate the available data in order to convey information about that data – often in the form of summaries or graphical displays, and usually for the purpose of simplifying comprehension or appreciation of the facts. These methods are concerned only with the available data, even if that data is only a part of a (possibly much larger) whole.
- ***Inferential methods*** use the available data in order to make general statements about the whole from which that data is only a part – i.e. they allow us to infer something about the whole from a study of just a part of it. Predictions and estimates are typical of the general conclusions which may be made by using inferential methods.

For example, a statistical analysis may be made of the ages of the eleven players in the England football team. The ages of the individual players form the data. If we determine the average age of the team from that data, that is applying descriptive methods to tell us something about the eleven players as a whole. However, the players are also a part of a much larger group – for example, all professional English football players – and clearly their average age will not necessarily be an accurate estimate of the average age of this larger group. If we wanted to draw conclusions about this larger group from the study of the eleven players in the England team, we would need to use inferential methods.

In this course, we shall only be concerned with descriptive statistical methods and, in this unit, we shall review the basis of the statistical method – the collection of facts (data) about a problem or issue – and then consider initial ways of analysing this data so that it has meaning in relation to the problem.

Objectives

When you have completed this study unit you will be able to:

- explain the concept of data in statistics;
- distinguish between primary and secondary data, and explain the differences between quantitative and qualitative variables and between continuous and discrete variables;
- explain the reasons for classifying and tabulating data, and prepare tables from given data;
- explain the concept of a frequency distribution and define the terms class interval, class boundary and class interval;
- prepare simple, grouped, cumulative and relative frequency distributions from given data.

A. DATA FOR STATISTICS

A statistical investigation involves a number of stages:

- definition of the problem or issue;
- collection of relevant data;
- classification and analysis of the collected data;
- presentation of the results.

Even before the collection of data starts, then, there are some important points to consider when planning a statistical investigation.

Preliminary Considerations

It is important to be aware of these issues as they impact on the data which is to be collected and analysed.

- **Exact definition of the problem**

  This is necessary in order to ensure that nothing important is omitted from the enquiry, and that effort is not wasted by collecting irrelevant data. The problem as originally put to the statistician is often of a very general type and it needs to be specified precisely before work can begin.

- **Definition of the units**

  The results must appear in comparable units for any analysis to be valid. If the analysis is going to involve comparisons, then the data must all be in the **same** units. It is no use just asking for "output" from several factories - some may give their answers in numbers of items, some in weight of items, some in number of inspected batches and so on.

- **Scope of the enquiry**

  No investigation should be got under way without defining the field to be covered. Are we interested in all departments of our business, or only some? Are we to concern ourselves with our own business only, or with others of the same kind?

- **Accuracy of the data**

  To what degree of accuracy is data to be recorded? For example, are ages of individuals to be given to the nearest year or to the nearest month or as the number of completed years? If some of the data is to come from measurements, then the accuracy of the measuring instrument will determine the accuracy of the results. The degree of precision required in an estimate might affect the amount of data we need to collect. In general, the more precisely we wish to estimate a value, the more readings we need to take.

Types of Data

There are a number of different ways in which data may be categorised.

- **Primary and secondary data**

  In its strictest sense, ***primary data*** is data which is both original and has been obtained in order to solve the specific problem in hand. Primary data is, therefore, raw data and has to be classified and processed using appropriate statistical methods in order to reach a solution to the problem.

  Secondary data is any data other than primary data. Thus, it includes any data which has been subject to the processes of classification or tabulation or which has resulted from the application of statistical methods to primary data, and all published statistics.

- **Quantitative and qualitative**

  Variables may be either quantitative or qualitative. ***Quantitative variables***, to which we shall restrict discussion here, are those for which observations are numerical in nature. ***Qualitative variables*** have non-numeric observations, such as colour of hair, although, of course, each possible non-numeric value may be associated with a numeric frequency.

- **Continuous and discrete**

  Variables may be either continuous or discrete. A ***continuous variable*** may take ***any value*** between two stated limits (which may possibly be minus and plus infinity). Height, for example, is a continuous variable, because a person's height may (with appropriately accurate equipment) be measured to any minute fraction of a millimetre. A ***discrete variable***, however, can take only ***certain values*** occurring at intervals between stated limits. For most (but not all) discrete variables, these interval values are the set of integers (whole numbers).

  For example, if the variable is the number of children per family, then the only possible values are 0, 1, 2, etc. because it is impossible to have other than a whole number of children. However, in Britain, shoe sizes are stated in half-units, and so here we have an example of a discrete variable which can take the values 1, 1½, 2, 2½, etc.

Requirements of Statistical Data

Having decided upon the preliminary matters about the investigation, the statistician must look in more detail at the actual data to be collected. The desirable qualities of statistical data are as follows.

- **Homogeneity**

  The data must be in properly comparable units. "Five houses" means little since five slum hovels are very different from five ancestral mansions. Houses cannot be compared unless they are of a similar size or value. If the data is found not to be homogeneous, there are two methods of adjustment possible.

  (a) Break down the group into smaller component groups which *are* homogeneous and study them separately.

  (b) Standardise the data. Use units such as "output per man-hour" to compare the output of two factories of very different size. Alternatively, determine a relationship between the different units so that all may be expressed in terms of one – in food consumption surveys, for example, a child may be considered equal to half an adult.

- **Completeness**

  Great care must be taken to ensure that no important aspect is omitted from the enquiry.

- **Accurate definition**

  Each term used in an investigation must be carefully defined; it is so easy to be slack about this and to run into trouble. For example, the term "accident" may mean quite different things to the injured party, the police and the insurance company! Watch out also, when using other people's statistics, for changes in definition. Acts of Parliament may, for example, alter the definition of an "indictable offence" or of an "unemployed person".

- **Uniformity**

  The circumstances of the data must remain the same throughout the whole investigation. It is no use, for example, comparing the average age of workers in an industry at two different times if the age structure has changed markedly. Likewise, it is not much use comparing a firm's profits at two different times if the working capital has changed.

B. METHODS OF COLLECTING DATA

When all the foregoing matters have been dealt with, we come to the question of how to collect the data we require. The methods usually available are as follows:

- Use of published statistics
- Personal investigation/interview
- Delegated personal investigation/interview
- Questionnaire.

Published Statistics

Sometimes we may be attempting to solve a problem that does not require us to collect new information, but only to reassemble and reanalyse data which has already been collected by someone else for some other purpose.

We can often make good use of the great amount of statistical data published by governments, the United Nations, nationalised industries, chambers of trade and commerce and so on. When using this method, it is particularly important to be clear on the definition of terms and units and on the accuracy of the data. The source must be reliable and the information up-to-date.

This type of data is sometimes referred to as ***secondary data*** in that the investigator himself has not been responsible for collecting it and it thus came to him "second-hand". By contrast, data which has been collected by the investigator for the particular survey in hand is called ***primary data***.

The information you require may not be found in one source but parts may appear in several different sources. Although the search through these may be time-consuming, it can lead to data being obtained relatively ***cheaply*** and this is one of the advantages of this type of data collection. Of course, the disadvantage is that you could spend a considerable amount of time looking for information which may not be available.

Another disadvantage of using data from published sources is that the definitions used for variables and units may not be the same as those you wish to use. It is sometimes difficult to establish the definitions from published information, but, before using the data, you must establish what it represents.

Personal Investigation/Interview

In this method the investigator collects the data him/herself. This method has the advantage that the data will be collected in a uniform manner and with the subsequent analysis in mind. However, the field that may be covered by personal investigation may, naturally, be limited. There is sometimes a danger that the investigator may be tempted to select data that accords with preconceived notions.

The personal investigation method is also useful if a pilot survey is carried out prior to the main survey, as personal investigation will reveal the problems that are likely to occur.

Delegated Personal Investigation/Interview

When the field to be covered is extensive, the task of collecting information may be too great for one person. Then a team of selected and trained investigators or interviewers may be used. The people employed should be properly trained and informed of the purposes of the investigation; their instructions must be very carefully prepared to ensure that the results are in accordance with the "requirements" described in the previous section of this study unit. If there are many investigators, personal biases may tend to cancel out.

Care in allocating the duties to the investigators can reduce the risks of bias. For example, if you are investigating the public attitude to a new drug in two towns, do not put investigator A to explore town X and investigator B to explore town Y, because any difference that is revealed might be due to the towns being different, *or* it might be due to different personal biases on the part of the two investigators. In such a case, you would try to get both people to do part of each town.

Questionnaire

In some enquiries the data consists of information which must be supplied by a large number of people. The most convenient way to collect such data is to issue questionnaire forms to the people concerned and ask them to fill in the answers to a set of printed questions. This method is usually cheaper than delegated personal investigation and can cover a wider field. A carefully thought-out questionnaire is often also used in the previous methods of investigation in order to reduce the effect of personal bias.

The distribution and collection of questionnaires by post suffers from two main drawbacks:

- The forms are completed by people who may be unaware of some of the requirements and who may place different interpretations on the questions – even the most carefully worded ones!
- There may be a large number of forms not returned, and these may be mainly by people who are not interested in the subject or who are hostile to the enquiry. The result is that we end up with completed forms only from a certain kind of person and thus have a biased sample. For this reason, it is essential to include a reply-paid envelope to encourage people to respond.

If the forms are distributed and collected by interviewers, a greater response is likely and queries can be answered. This is the method used, for example, in the UK Population Census. Care must be taken, however, that the interviewers do not lead respondents in any way.

C. CLASSIFICATION AND TABULATION OF DATA

Having completed the survey and collected the data, we need to organise it so that we can extract useful information and then present our results.

The data will very often consist of a mass of figures in no very special order. For example, we may have a card index of the 3,000 workers in a large factory; the cards are probably kept in alphabetical order of names, but they will contain a large amount of other data such as wage rates, age, sex, type of work, technical qualifications and so on. If we are required to present to the factory management a statement about the age structure of the labour force (both male and female), then the alphabetical arrangement does not help us, and no one could possibly gain any idea about the topic from merely looking through the cards as they are.

What is needed is to ***classify*** the cards according to the age and sex of the worker and then present the results of the classification as a ***tabulation***.

The data in its original form, before classification, is usually known as "***raw data***".

Example

We cannot, of course, give here an example involving 3,000 cards, but you ought now to follow this "shortened version" involving only a small number of items.

Let us assume that our raw data about workers consists of 15 cards in alphabetical order, as follows:

Archer, L. Mr	39 years
Black, W. Mrs	20 ″
Brown, A. Mr	22 ″
Brown, W. Miss	22 ″
Carter, S. Miss	32 ″
Dawson, T. Mr	30 ″

Dee, C. Mrs	37	″
Edwards, D. Mr	33	″
Edwards, R. Mr	45	″
Gray, J. Mrs	42	″
Green, F. Miss	24	″
Gunn, W. Mr	27	″
Jackson, J. Miss	28	″
Johnson, J. Mr	44	″
Jones, L. Mr	39	″

As it stands, this data does not allow us to draw any useful conclusions about the group of workers. In order to do this, we need to classify the data in some way. One way would be to classify the individual cases by sex:

Archer, L. Mr	39 years	Black, W. Mrs	20 years
Brown, A. Mr	22 ″	Brown, W. Miss	22 ″
Dawson, T. Mr	30 ″	Carter, S. Miss	32 ″
Edwards, D. Mr	33 ″	Dee. C. Mrs	37 ″
Edwards, R. Mr	45 ″	Gray, J. Mrs	42 ″
Gunn, W. Mr	27 ″	Green, F. Miss	24 ″
Johnson, J. Mr	44 ″	Jackson, J. Miss	28 ″
Jones, L. Mr	39 ″		

An alternative way of classifying the group would be by age, splitting them into three groups as follows:

Age between 20 and 29		***Age between 30 and 39***	
Black, W. Mrs	20 years	Archer, L. Mr	39 years
Brown, A. Mr	22 ″	Carter, S. Miss	32 ″
Brown, W. Miss	22 ″	Dawson, T. Mr	30 ″
Green, F. Miss	24 ″	Dee, C. Mrs	37 ″
Gunn, W. Mr	27 ″	Edwards, D. Mr	33 ″
Jackson, J. Miss	28 ″	Jones, L. Mr	39 ″

Age over 40	
Edwards, R. Mr	45 years
Gray, J. Mrs	42 ″
Johnson, J. Mr	44 ″

Again, although the data has now acquired some of the characteristics of useful information, it is still a collection of data about individuals. In statistics, though, we are concerned with things in groups rather than with individuals. In comparing, say, the height of Frenchmen with the height of Englishmen, we are concerned with Frenchmen in general and Englishmen in general, but not with

Marcel and John as individuals. An insurance company, to give another example, is interested in the proportion of men (or women) who die at certain ages, but it is not concerned with the age at which John Smith (or Mary Brown), as individuals, will die.

We need to transform the data into information about the group, and we can do this by tabulation. The number of cards in each group, after classification, is counted and the results presented in a table.

Table 6.1: Workers by age and sex

Age group	Sex		Total
	Male	Female	
20 – 29	2	4	6
30 – 39	4	2	6
40 – 49	2	1	3
Total	8	7	15

You should look through this example again to make quite sure that you understand what has been done.

You are now in a position to appreciate the purpose behind classification and tabulation - it is to condense an unwieldy mass of raw data to manageable proportions and then to present the results in a readily understandable form.

Forms of Tabulation

We can classify the process of tabulation into simple tabulation and complex or matrix tabulation.

(a) Simple Tabulation

This covers only one aspect of the set of figures. The idea is best conveyed by an example.

Consider the card index of 3,000 workers mentioned earlier. Assume that each card carries the name of the workshop in which the person works. A question as to how the labour force is distributed can be answered by sorting the cards and preparing a simple table as follows:

Table 6.2: Distribution of workers by workshop

Workshop	Number Employed
A	600
B	360
C	660
D	840
E	540
Total	3,000

Another question might have been, “What is the wage distribution in the works?”, and the answer can be given in another simple table (see Table 7).

Table 6.3: Wage distribution

Wages Group	Number of Employees
£40 *but less than* £60	105
£60 *but less than* £80	510
£80 *but less than* £100	920
£100 *but less than* £120	1,015
£120 *but less than* £140	300
£140 *but less than* £160	150
Total	3,000

Note that such simple tables do not tell us very much - although it may be enough for the question of the moment.

(b) Complex tabulation

This deals with two or more aspects of a problem at the same time. In the problem just studied, it is very likely that the two questions would be asked at the same time, and we could present the answers in a complex table or ***matrix***.

Table 6.4: Distribution of workers and wages by workshop

Workshop	Wages Group *(£ per week)*						Total no. employed
	40 – 59.99*	60 – 79.99	80 – 99.99	100 – 119.99	120 – 139.99	140 – 159.99	
A	20	101	202	219	29	29	600
B	11	52	90	120	29	58	360
C	19	103	210	200	88	40	660
D	34	167	303	317	18	1	840
E	21	87	115	159	136	22	540
Total	105	510	920	1,015	300	150	3,000

Note: *40 – 59.99 is the same as "40 but less than 60" and similarly for the other columns.

This table is much more informative than are the two previous simple tables, but it is more complicated. We could have complicated it further by dividing the groups into, say, male and female workers, or into age groups.

Later in the unit, we will look at a list of the rules you should try to follow in compiling statistical tables, and at the end of that list you will find a table relating to our 3,000 workers, which you should study as you read the rules.

Secondary Statistical Tabulation

So far, our tables have merely classified the already available figures – the primary statistics. However, we can go further than this and do some simple calculations to produce other figures – secondary statistics.

As an example, take the first simple table illustrated above, and calculate how many employees there are on average per workshop. This is obtained by dividing the total (3,000) by the number of shops (5), and the table appears as follows:

Table 6.5: Distribution of workers by workshop

Workshop	Number Employed
A	600
B	360
C	660
D	840
E	540
Total	3,000
Average number of employees per workshop:	600

The average is a "secondary statistic".

For another example, we may take the second simple table given above and calculate the proportion of workers in each wage group, as follows:

Table 6.6: Wage distribution

Wages Group	Number of Employees	Proportion of Employees
£40 – £60	105	0.035
£60 – £80	510	0.170
£80 – £100	920	0.307
£100 – £120	1,015	0.338
£120 – £140	300	0.100
£140 – £160	150	0.050
Total	3,000	1.000

The proportions given here are also secondary statistics. In commercial and business statistics, it is more usual to use percentages than proportions – so in the above table, these would be 3.5%, 17%, 30.7%, 33.8%, 10% and 5%.

Secondary statistics are not, of course, confined to simple tables, they are used in complex tables too, as in this example:

Table 6.7: Inspection results for a factory product in two successive years

Machine No.	Year 1			Year 2		
	Output	No. of Rejects	% of Rejects	Output	No. of Rejects	% of Rejects
1	800	40	5.0	1,000	100	10.0
2	600	30	5.0	500	100	20.0
3	300	12	4.0	900	45	5.0
4	500	10	2.0	400	20	5.0
Total	2,200	92	4.2	2,800	265	9.5
Average per machine:	550	23	4.2	700	66.2	9.5

The percentage columns and the average line show secondary statistics. All the other figures are primary statistics.

Note carefully that ***percentages cannot be added or averaged to get the percentage of a total or of an average***. You must work out such percentages on the ***totals or averages themselves***.

Another danger in the use of percentages has to be watched, and that is that you must not forget the size of the original numbers. Take, for example, the case of two doctors dealing with a certain disease. One doctor has only one patient and he cures him – 100% success! The other doctor has 100 patients of whom he cures 80 – only 80% success! You can see how very unfair it would be on the hard-working second doctor to compare the percentages alone.

Rules for Tabulation

There are no absolute rules for drawing up statistical tables, but there are a few general principles which, if borne in mind, will help you to present your data in the best possible way.

- Try not to include too many features in any one table (say, not more than four or five) as otherwise it becomes rather clumsy. It is better to use two or more separate tables.
- Each table should have a clear and concise title to indicate its purpose.
- It should be very clear what units are being used in the table (tonnes, £, people, £000, etc.).
- Blank spaces and long numbers should be avoided, the latter by a sensible degree of approximation.
- Columns should be numbered to facilitate reference.
- Try to have some order to the table, using, for example, size, time, geographical location or alphabetical order.
- Figures to be compared or contrasted should be placed as close together as possible.
- Percentages should be pleased near to the numbers on which they are based.
- Rule the tables neatly – scribbled tables with freehand lines nearly always result in mistakes and are difficulty to follow. However, it is useful to draw a rough sketch first so that you can choose the best layout and decide on the widths of the columns.

- Insert totals where these are meaningful, but avoid "nonsense totals". Ask yourself what the total will tell you before you decide to include it. An example of such a "nonsense total" is given in the following table:

Table 6.8: Election Results

Party	Year 1 Seats	Year 2 Seats	Total Seats
A	350	200	550
B	120	270	390
Total	470	470	940

The totals (470) at the foot of the two columns make sense because they tell us the total number of seats being contested, but the totals in the final column (550, 390, 940) are "nonsense totals" for they tell us nothing of value.

- If numbers need to be totalled, try to place them in a column rather than along a row for easier computation.
- If you need to emphasise particular numbers, then underlining, significant spacing or heavy type can be used. If data is lacking in a particular instance, then insert an asterisk (*) in the empty space and give the reasons for the lack of data in a footnote.
- Footnotes can also be used to indicate, for example, the source of secondary data, a change in the way the data has been recorded, or any special circumstances which make the data seem odd.

It is not always possible to obey all of these rules on any one occasion, and there may be times when you have a good reason for disregarding some of them. But only do so if the reason is really good – not just to save you the bother of thinking!

Study now the layout of the following table (based on our previous example of 3,000 workpeople) and check through the list of rules to see how they have been applied.

Table 6.9: ABC & Co. – Wage structure of labour force
Numbers of persons in specified categories

Workshop	Wage Group *(£ per week)*						Total no. employed	% *See note (a)*
	40 – 59.99	60 – 79.99	80 – 99.99	100 – 119.99	120 – 139.99	140 - 159.99		
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*
A	20	101	202	219	29	29	600	20.0
B	11	52	90	120	29	58	360	12.0
C	19	103	210	200	88	40	660	22.0
D	34	167	303	317	18	1	840	28.0
E	21	87	115	159	136	22	540	18.0
Total no. in wage group	105	510	920	1,015	300	150	3,000	100.0
% *See note (b)*	3.5	17.0	30.7	33.8	10.0	5.0	100.0	–

Note (a): Total number employed in each workshop as a percentage of the total workforce.

Note (b): Total number in each wage group as a percentage of the total workforce.

This table can be called a "twofold" table as the workforce is broken down by wage and workshop.

Practice Questions 1

These questions are typical of those likely to set in an examination.

1. The personnel manager of your firm submits the following report on labour and staff to your managing director:

   > "The total strength is 136, of whom 62 are male. Of these men, 43 are industrial labour and the rest are staff. The age distribution of the 43 is "under 18" (seven), "18-45" (22) and the others are over 45. Our total industrial labour force is 113, of whom 25 are under 18 and 19 are over 45. The total number of under-18s on the site is 29 (10 male, 19 female). There are 23 over-45s, of whom only 4 are male staff."

   The managing director, quite rightly, thinks that this is gibberish and he instructs you to present it to him in a tabular form which he can readily understand. Show the table you would produce.

2. "In the period Year 1 – Year 6 there were 1,775 stoppages of work arising from industrial disputes on construction sites. Of these, 677 arose from pay disputes, 111 from conflict over demarcation, 144 on working conditions and 843 were unofficial walk-outs. 10% of all stoppages lasted only one day. Of these, the corresponding proportions were: 11% for pay disputes, 6% in the case of demarcation and 12½% in the case of working conditions.

   In the subsequent period Year 7 – Year 12 the number of stoppages was 2,807 higher. But, whereas the number of stoppages arising from pay disputes fell by 351 and those concerning working conditions by 35, demarcation disputes rose by 748. The number of pay disputes resolved within one day decreased by 45 compared with the earlier period, but only by 2 in the case of working conditions. During the later period 52 stoppages over demarcation were settled within the day, but the number of stoppages from all causes which lasted only one day was 64 greater in the later period than in the earlier."

You are required to:

(a) Tabulate the information in the above passage, making whatever calculations are needed to complete the entries in the table.

(b) Write a brief comment on the main features disclosed by the completed table.

Now check your answers with the ones given at the end of the unit.

D. FREQUENCY DISTRIBUTIONS

A frequency distribution is a tabulation which shows the number of times (i.e. the frequency) each different value occurs.

For example, the following figures are the times (in minutes) taken by a shop-floor worker to perform a given repetitive task on 20 specified occasions during the working day:

3.5	3.8	3.8	3.4	3.6
3.6	3.8	3.9	3.7	3.5
3.4	3.7	3.6	3.8	3.6
3.7	3.7	3.7	3.5	3.9

If we now assemble and tabulate these figures, we can obtain a frequency distribution as follows:

Table 6.10: Time taken for one worker to perform task

Length of time (*minutes*)	Frequency
3.4	2
3.5	3
3.6	4
3.7	5
3.8	4
3.9	2
Total	20

Simple Frequency Distributions

A useful way of preparing a frequency distribution from raw data is to go through the records as they stand and mark off the items by the "tally mark" or "five-bar gate" method. First look at the figures to see the highest and lowest values so as to decide the range to be covered and then prepare a blank table. Then mark the items on your table by means of a tally mark.

To illustrate the procedure, the following table shows the procedure for the above example after all 20 items have been entered.

Table 6.11: Time taken for one worker to perform task

<table>
<tr><th>Length of time
(minutes)</th><th>Tally marks</th><th>Frequency</th></tr>
<tr><td>3.4</td><td>| |</td><td>2</td></tr>
<tr><td>3.5</td><td>| | |</td><td>3</td></tr>
<tr><td>3.6</td><td>| | | |</td><td>4</td></tr>
<tr><td>3.7</td><td><s>| | | |</s></td><td>5</td></tr>
<tr><td>3.8</td><td>| | | |</td><td>4</td></tr>
<tr><td>3.9</td><td>| |</td><td>2</td></tr>
<tr><td>Total</td><td></td><td>20</td></tr>
</table>

Grouped Frequency Distributions

Sometimes the data is so extensive that a simple frequency distribution is too cumbersome and, perhaps, uninformative. Then we make use of a "***grouped frequency distribution***".

In this case, the "length of time" column consists not of separate values, but of groups of values:

Table 6.12: Time taken for one worker to perform task

<table>
<tr><th>Length of time
(minutes)</th><th>Tally marks</th><th>Frequency</th></tr>
<tr><td>3.4 to 3.5</td><td><s>| | | |</s></td><td>5</td></tr>
<tr><td>3.6 to 3.7</td><td><s>| | | |</s> | | | |</td><td>9</td></tr>
<tr><td>3.8 to 3.9</td><td><s>| | | |</s> |</td><td>6</td></tr>
<tr><td>Total</td><td></td><td>20</td></tr>
</table>

Grouped frequency distributions are only needed when there is a large number of values and, in practice, would not have been required for the small amount of data in our example. Table 6.13 shows a grouped frequency distribution used in a more realistic situation, when an ungrouped table would not have been of much use.

Table 6.13: Age distribution of workers in an office

Age group *(years)*	Number of workers
16 – 20	10
21 – 25	17
26 – 30	28
31 – 35	42
36 – 40	38
41 – 45	30
46 – 50	25
51 – 55	20
56 – 60	10
Total	220

The various groups (for example, “26 – 30”) are called “***classes***” and the range of values covered by a class (five years in this example) is called the “***class interval***”. The number of items in each class (for example, 28 in the 26 – 30 class) is called the “***class frequency***” and the total number of items (in this example, 220) is called the “***total frequency***”.

The term “***class boundary***” is used to denote the dividing line between adjacent classes, so in the age group example the class boundaries are 16, 21, 26, ... years. In the length of time example, as grouped earlier in this section, the class boundaries are 3.35, 3.55, 3.75, 3.95 minutes. This needs some explanation. As the original readings were given correct to one decimal place, we assume that is the precision to which they were measured. If we had had a more precise stopwatch, the times could have been measured more precisely. In the first group of 3.4 to 3.5 are put times which could in fact be anywhere between 3.35 and 3.55 if we had been able to measure them more precisely. A time such as 3.57 minutes would not have been in this group as it equals 3.6 minutes when corrected to one decimal place and it goes in the 3.6 to 3.7 group.

Another term, “***class limits***”, is used to stand for the lowest and highest values that can actually occur in a class. In the age group example, these would be 16 years and 20 years 364 days for the first class, 21 years and 25 years 364 days for the second class and so on, assuming that the ages were measured correct to the nearest day below. In the length of time example, the class limits are 3.4 and 3.5 minutes for the first class and 3.6 and 3.7 minutes for the second class.

You should make yourself quite familiar with these terms.

Choice of Class Interval

When compiling a frequency distribution you should, if possible, make the length of the class interval equal for all classes so that fair comparison can be made between one class and another. Sometimes, however, this rule has to be broken – for example, some grouped frequency distributions bring together the first few classes and the last few classes because there are few instances in each of the individual classes.

In such cases, before using the information, it is as well to make the classes comparable by calculating a column showing “frequency per interval of so much”, as in this example for wage statistics:

Table 6.14: Annual income statistics

Annual income *(£)*	No. of persons	Frequency per £200 interval
4,801 – 5,000	50,000	50,000
5,001 – 5,200	40,000	40,000
5,201 – 5,600	55,000	27,500
5,601 – 6,000	23,000	11,500
6,001 – 6,400	12,000	6,000
6,401 – 7,200	12,000	3,000
Total	192,000	–

Notice that the intervals in the first column are:

200, 200, 400, 400, 400, 800.

These intervals let you see how the last column was compiled.

A superficial look at the original table (first two columns only) might have suggested that the most frequent incomes were at the middle of the scale, because of the appearance of the figure 55,000. But this apparent preponderance of the middle class is due solely to the change in the length of the class interval, and column three shows that, in fact, the most frequent incomes are at the bottom end of the scale, i.e. the top of the table.

You should remember that the purpose of compiling a grouped frequency distribution is to make sense of an otherwise troublesome mass of figures. It follows, therefore, that we do not want to have too many groups or we will be little better off; nor do we want too few groups or we will fail to see the significant features of the distribution. As a practical guide, you will find that somewhere between about five and 10 groups will usually be suitable.

Cumulative Frequency Distributions

Very often we are not specially interested in the separate class frequencies, but in the number of items above or below a certain value. When this is the case, we form a ***cumulative frequency distribution*** as illustrated in column three of the following table:

Table 6.15: Time taken for one worker to perform task

Length of time *(minutes)*	Frequency	Cumulative frequency
3.4	2	2
3.5	3	5
3.6	4	9
3.7	5	14
3.8	4	18
3.9	2	20

The cumulative frequency tells us the number of items equal to or less than the specified value, and it is formed by the successive addition of the separate frequencies. A cumulative frequency column may also be formed for a grouped distribution.

The above example gives us the number of items "less than" a certain amount, but we may wish to know, for example, the number of persons having ***more than*** some quantity. This can easily be done by doing the cumulative additions from the bottom of the table instead of the top, and as an exercise you should now compile the "more than" cumulative frequency column in the above example.

Relative Frequency Distributions

All the frequency distributions which we have looked at so far in this study unit have had their class frequencies expressed simply as numbers of items. However, remember that proportions or percentages are useful secondary statistics. When the frequency in each class of a frequency distribution is given as a proportion or percentage of the total frequency, the result is known as a "***relative frequency distribution***" and the separate proportions or percentages are the "***relative frequencies***".

The total relative frequency is, of course, always 1.0 (or 100%). Cumulative relative frequency distributions may be compiled in the same way as ordinary cumulative frequency distributions.

As an example, the distribution used in Table 6.14 is now set out as a relative frequency distribution for you to study.

Table 6.16: Annual income statistics

Annual income *(£)*	Frequency	Relative frequency (%)	Cumulative frequency	Relative cumulative frequency (%)
4,801 – 5,000	50,000	26	50,000	26
5,001 – 5,200	40,000	21	90,000	47
5,201 – 5,600	55,000	29	145,000	76
5,601 – 6,000	23,000	12	168,000	88
6,001 – 6,400	12,000	6	180,000	94
6,401 – 7,200	12,000	26	192,000	100
Total	192,000	100	–	–

This example is in the "less than" form, and you should now compile the "more than" form in the same way as you did for the non-relative distribution.

Practice Questions 2

The following figures show the number of faulty items made on successive days by a machine in an engineering workshop:

2	0	2	4	3	2	11	4	0	1
1	4	1	2	1	4	1	7	3	2
4	5	5	6	5	2	4	2	1	2
2	4	0	6	1	0	1	5	1	2
3	1	5	3	2	2	1	3	5	3
1	1	10	1	2	3	4	3	10	6
2	6	5	3	1	2	5	7	2	3
4	4	1	3	7	4	1	7	3	7
1	1	2	6	1	0	8	5	8	1
5	6	1	3	2	2	1	2	6	4

Tabulate these figures as a frequency distribution, giving columns containing the frequency, the relative frequency, the cumulative frequency and the relative cumulative frequency

Now check your answers with the ones given at the end of the unit.

ANSWERS TO QUESTIONS FOR PRACTICE

Practice Questions 1

1. ***"XYZ" Company: Age of Staff and Labour***

Age group	Industrial			Staff			Total		
	Male	Female	Total	Male	Female	Total	Male	Female	Total
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*
Under 18	7	18	25	3	1	4	10	19	29
18 – 45	22	47	69	12	3	15	34	50	84
Over 45	14	5	19	4	0	4	18	5	23
TOTAL	43	70	113	19	4	23	62	74	136

2. (a) The complete table is as follows:

Number of Stoppages of Work Arising from Industrial Disputes on Construction Sites

Cause	Year 1 – Year 6			Year 7 – Year 12		
	1 day	More than 1 day	Total	1 day	More than 1 day	Total
Pay	74.47	602.53	677	29.47	296.53	326
Demarcation	6.66	104.34	111	52	807	859
Working conditions	18	126	144	16	93	109
Unofficial	78.37	764.63	843	144.03	3,143.97	3,288
All	177.5	1,597.5	1,775	241.5	4,340.5	4,582

Note that we have given the figures as worked out from the information supplied. Obviously, though, fractional numbers of stoppages cannot occur, and you should round off each figure to the nearest whole number.

The sub-totals of the columns will then be 177, 1,598, 241, 4,341.

(b) The overall number of stoppages in the later period was more than 2½ times greater than in the earlier period. In both periods, the number of unofficial stoppages was the highest category and this was particularly remarkable in the later period, when they amounted to more than twice the number of all other stoppages put together. Demarcation disputes rose in number dramatically, although stoppages over pay and working conditions both declined. About 95% of stoppages in the later period lasted more than one day, compared with 90% in the previous period, with those concerning working conditions being less likely to last more than one day.

Practice Questions 2

1. ***Faulty items made on successive days by a machine in an engineering workshop***

Number of faulty items	Number of days with these faulty items	Relative frequency	Cumulative frequency	Relative cumulative frequency
0	5	0.05	5	0.05
1	23	0.23	28	0.28
2	20	0.20	48	0.48
3	13	0.13	61	0.61
4	12	0.12	73	0.73
5	10	0.10	83	0.83
6	7	0.07	90	0.90
7	5	0.05	95	0.95
8	2	0.02	97	0.97
9	0	0.00	97	0.97
10	2	0.02	99	0.99
11	1	0.01	100	1.00
Total	100	1.00	–	–

Study Unit 7

Graphical Representation of Information

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INTRODUCTION

In this unit we shall move on from the previous unit to look at further methods of summarising and presenting data to provide useful information.

Diagrams are particularly effective as a means of conveying quite complex information. Presenting information visually is easy to understand and enables broad distributions and trends to be taken in quickly. However, they are rarely as accurate as the figures themselves and details are likely to be lost.

In the first part of the unit we shall examine the specific graphical forms of presenting frequency distributions of the type considered in the previous unit. We then move on to discuss a number of common graphical presentations that are designed more for the lay reader than someone with statistical knowledge. You will certainly have seen some examples of them used in the mass media of newspapers and television.

Objectives

When you have completed this study unit you will be able to:

- use frequency dot diagrams, frequency bar charts, histograms and ogives to represent frequency distributions;
- prepare pictograms, circular diagrams and bar charts from given data, and discuss the circumstances in which each form of presentation is most effective;
- describe the use of Lorenz curves and prepare such curves from given data;
- describe the general rules and principles for using graphical representations of data.

A. GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTIONS

Tabulated frequency distributions are sometimes more readily understood if represented by a diagram. Graphs and charts are normally much superior to tables (especially lengthy complex tables) for showing general states and trends, but they cannot usually be used for accurate analysis of data.

The methods of presenting frequency distributions graphically are as follows:

- Frequency dot diagram
- Frequency bar chart
- Frequency polygon
- Histogram
- Ogive.

We will now examine each of these in turn, initially using the data on the times (in minutes) taken by a worker to perform a given task on 20 specified occasions during the working day which we considered in the last unit. The tabulated frequency distribution for this data is as follows:

Table 7.1: Time taken for one worker to perform task

Length of time (*minutes*)	Frequency
3.4	2
3.5	3
3.6	4
3.7	5
3.8	4
3.9	2
Total	20

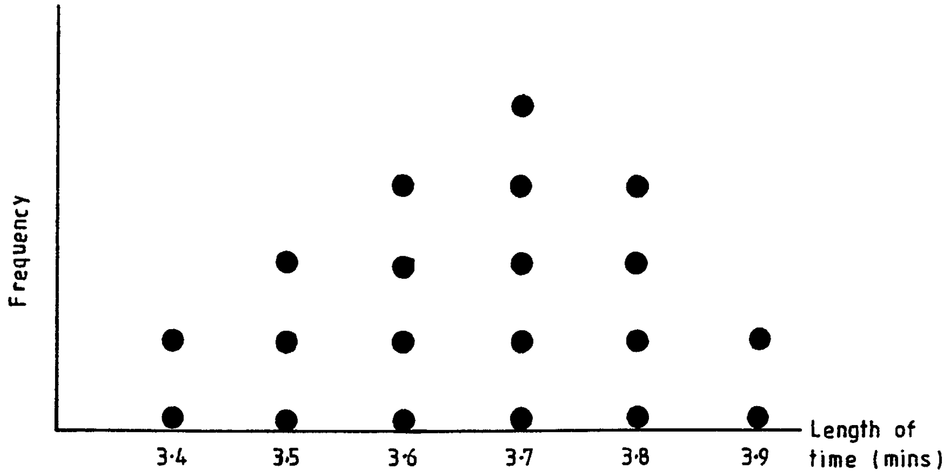
Frequency Dot Diagram

This is a simple form of graphical representation for the frequency distribution of a ***discrete*** variate.

A horizontal scale is used for the variate and a vertical scale for the frequency. Above each value on the variate scale we mark a dot for each occasion on which that value occurs.

Thus, a frequency dot diagram of the information in Table 7.1 would be as shown in Figure 7.1.

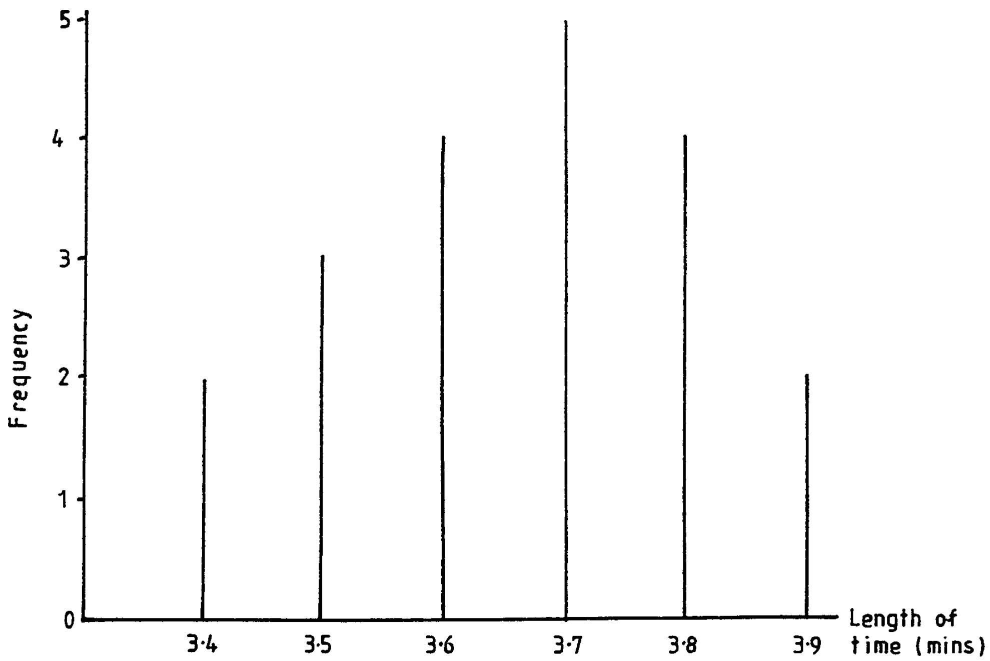
Figure 7.1: Frequency dot diagram to show length of time taken by operator to complete a given task



Frequency Bar Chart

We can avoid the business of marking every dot in such a diagram by drawing instead a vertical line the length of which represents the number of dots which should be there. The frequency dot diagram in Figure 7.1 now becomes a frequency bar chart, as in Figure 7.2.

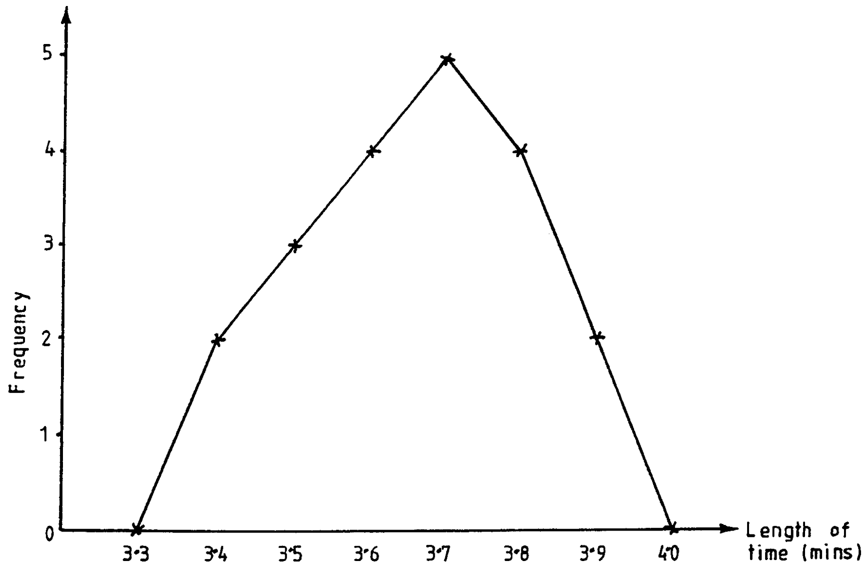
Figure 7.2: Frequency bar chart to show length of time taken by operator to complete a given task



Frequency Polygon

Instead of drawing vertical bars as we do for a frequency bar chart, we could merely mark the position of the top end of each bar and then join up these points with straight lines. When we do this, the result is a frequency polygon, as in Figure 7.3.

Figure 7.3: Frequency polygon to show length of time taken by operator to complete a given task



Note that we have added two fictitious classes at each end of the distribution, i.e. we have marked in groups with zero frequency at 3.3 and 4.0. This is done to ensure that the area enclosed by the polygon and the horizontal axis is the same as the area under the corresponding ***histogram*** which we shall consider in the next section.

These three kinds of diagram are all commonly used as a means of making frequency distributions more readily comprehensible. They are mostly used in those cases where the variate is ***discrete*** and where the values are ***not grouped***. Sometimes, frequency bar charts and polygons are used with grouped data by drawing the vertical line (or marking its top end) at the centre point of the group.

Histogram

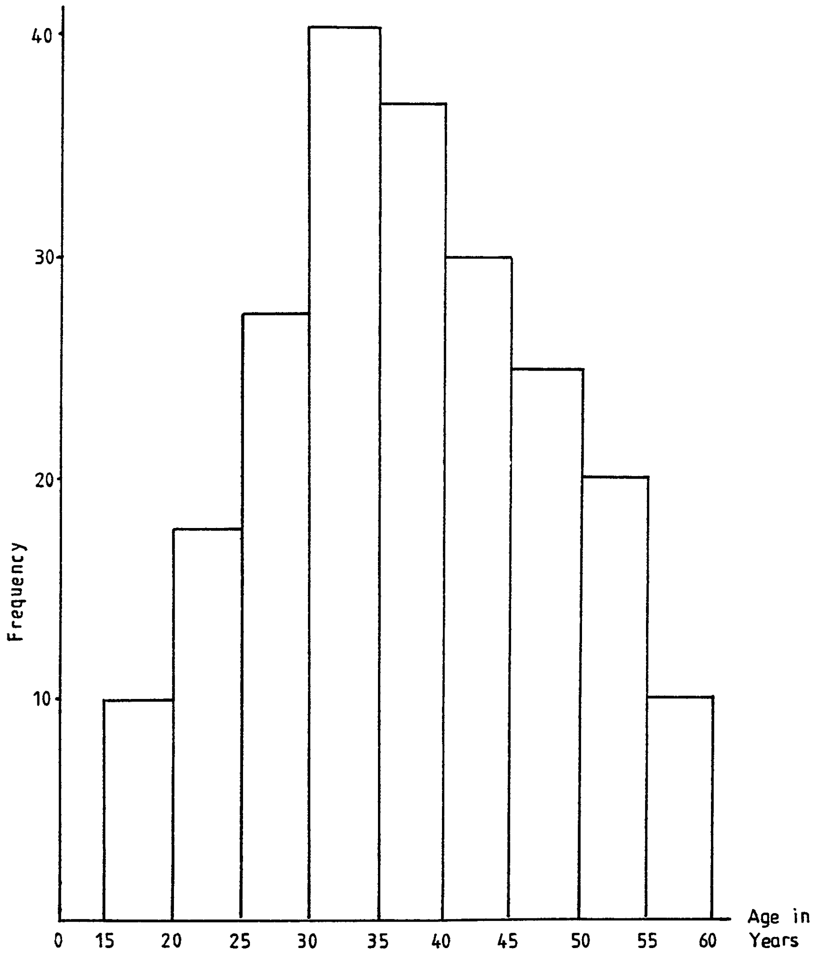
This is the best way of showing a grouped frequency distribution.

To illustrate this, we shall use the example of a grouped frequency distribution from the previous unit as set out in Table 7.2. The information in the table may be shown graphically by the histogram in Figure 7.4.

Table 7.2: Age distribution of workers in an office

Age group *(years)*	Number of workers
16 – 20	10
21 – 25	17
26 – 30	28
31 – 35	42
36 – 40	38
41 – 45	30
46 – 50	25
51 – 55	20
56 – 60	10
Total	220

Figure 7.4: Histogram showing age distribution of workers in an office



In a histogram, the frequency in each group is represented by a rectangle and – this is a very important point – it is the ***area of the rectangle, not its height, which represents the frequency***.

When the lengths of the class intervals are all equal, then the heights of the rectangles represent the frequencies in the same way as do the areas (this is why the vertical scale has been marked in this diagram). If, however, the lengths of the class intervals are not all equal, you must remember that the heights of the rectangles have to be adjusted to give the correct areas. (Do not stop at this point if you have not quite grasped the idea, because it will become clearer as you read on.)

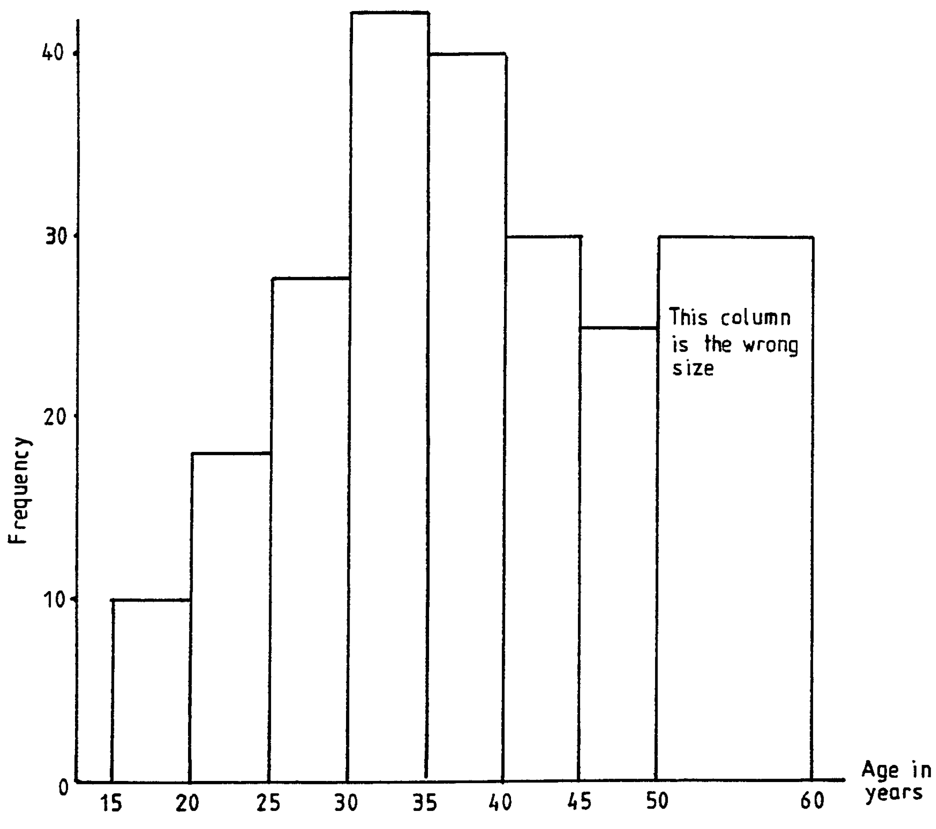
Look once again at the histogram of ages given in Figure 7.4 and note particularly how it illustrates the fact that the frequency falls off towards the higher age groups – any form of graph which did not reveal this fact would be misleading. Now let us imagine that the original table had *not* used equal class intervals but, for some reason or other, had given the last few groups as:

Table 7.3: Age distribution of workers in an office (amended extract)

Age group *(years)*	Number of workers
41 – 45	30
46 – 50	25
51 – 60	30

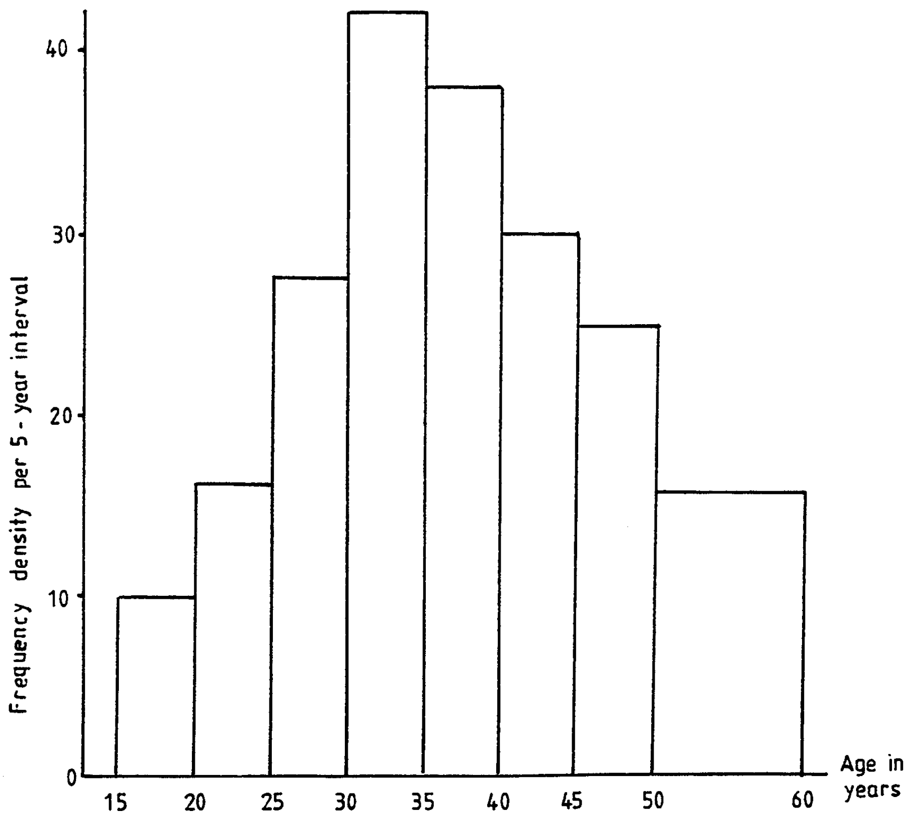
The last two groups have been lumped together as one. A ***wrong*** form of histogram, using heights instead of areas, for the amended data would look like Figure 7.5.

Figure 7.5: Histogram showing age distribution of workers in an office (amended data)



Now, this clearly gives an entirely wrong impression of the distribution with respect to the higher age groups. In the correct form of the histogram, the height of the last group (50 – 60) would be halved because the class interval is double all the other class intervals. The histogram in Figure 7.6 gives the right impression of the falling off of frequency in the higher age groups. The vertical axis has been labelled “Frequency density per 5-year interval” as five years is the “standard” interval on which we have based the heights of our rectangles.

Figure 7.6: Amended histogram showing age distribution of workers in an office (amended data)



Often it happens, in published statistics, that the last group in a frequency table is not completely specified. For example, the last few groups in our age distribution table may have been specified as follows:

Table 7.4: Age distribution of workers in an office (further amended extract)

Age group *(years)*	Number of workers
41 – 45	30
46 – 50	25
51 and over	30

A distribution of this kind is called an “***open-ended***” distribution.

How would we draw this on a histogram?

If the last (open-ended) group has a very small frequency compared with the total frequency (say, less than about 1% or 2%), then nothing much is lost by leaving it off the histogram altogether. If the last group has a larger frequency than about 1% or 2%, then you should try to judge from the general shape of the histogram how many class intervals to spread the last frequency over in order not to create a false impression of the extent of the distribution. In the example given, you would probably spread the last 30 people over two or three class intervals, but it is often simpler to assume that an open-ended class has the same length as its neighbour.

Whatever procedure is adopted, it is essential to state clearly what you have done and why.

The Ogive

This is the name given to the graph of the cumulative frequency. It can be drawn in either the "less than" or the "or more" form, but the "less than" form is the usual one. Ogives for two of the distributions already considered in course are now given as examples; Figure 7.7 is for ungrouped data (as per Table 7.5) and Figure 7.8 is for grouped data (as per Table 7.6).

Study these two diagrams so that you are quite sure that you know how to draw them. There is only one point which you might be tempted to overlook in the case of the grouped distribution – the points are plotted at the ***ends*** of the class intervals and *not* at the centre point. Look at the example and see how the 168,000 is plotted against the ***upper end*** of the 56 – 60 group and not against the mid-point, 57.5. If we had been plotting an "or more" ogive, the plotting would have to have been against the lower end of the group.

Table 7.5: Time taken for one worker to perform task

Length of time *(minutes)*	Frequency	Cumulative frequency
3.4	2	2
3.5	3	5
3.6	4	9
3.7	5	14
3.8	4	18
3.9	2	20

Table 7.6: Annual income statistics

Annual income *(£)*	Frequency	Cumulative frequency
4,801 – 5,000	50,000	50,000
5,001 – 5,200	40,000	90,000
5,201 – 5,600	55,000	145,000
5,601 – 6,000	23,000	168,000
6,001 – 6,400	12,000	180,000
6,401 – 7,200	12,000	192,000
Total	192,000	–

Figure 7.7: Ogive to show length of time taken by operator to complete a given task

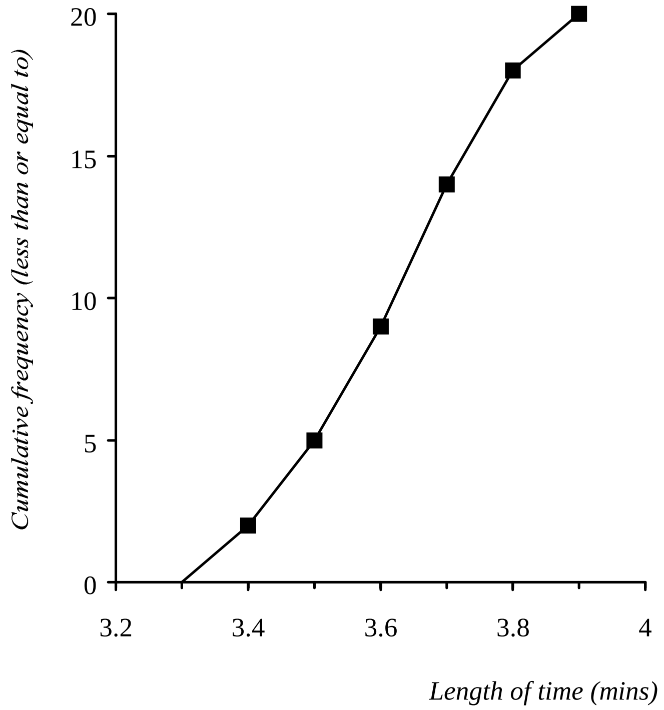
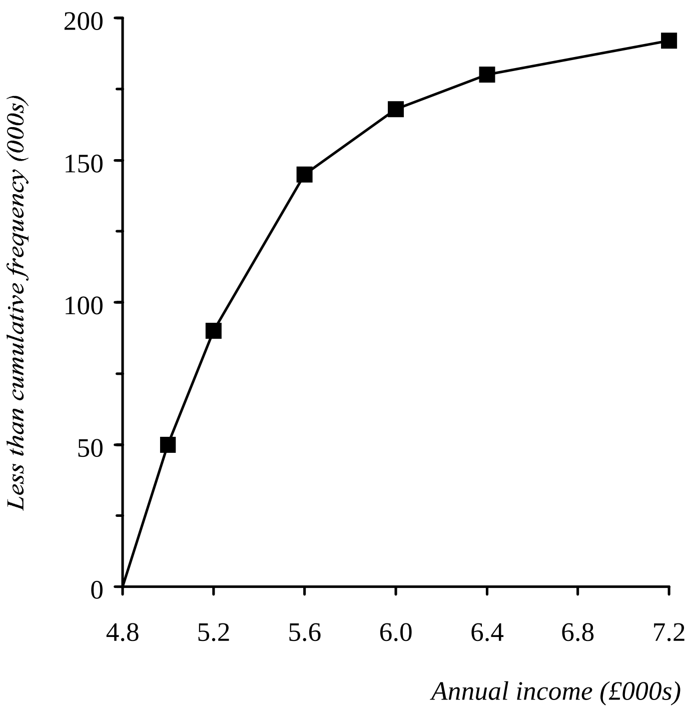


Figure 7.8: Ogive to show annual income distribution



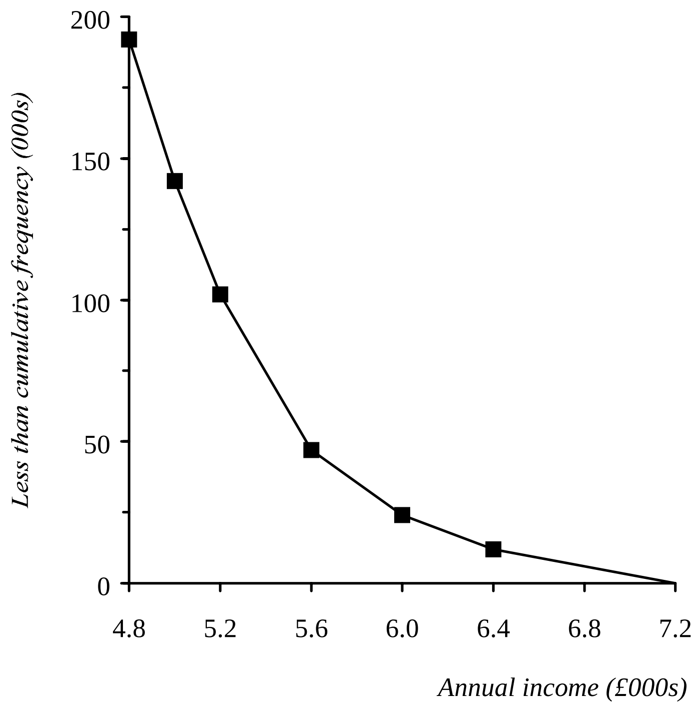
As an example of an "or more" ogive, we can compile the data from our example of annual income statistics as a "more than" cumulative frequency:

Table 7.7: Annual income statistics

Annual income *(£)*	Frequency	Cumulative frequency *(more than)*
4,801 – 5,000	50,000	192,000
5,001 – 5,200	40,000	142,000
5,201 – 5,600	55,000	102,000
5,601 – 6,000	23,000	47,000
6,001 – 6,400	12,000	24,000
6,401 – 7,200	12,000	12,000
Total	192,000	–

The ogive for this now appears as shown in Figure 7.9.

Figure 7.8: Ogive to show annual income distribution



Check that you see how the plotting has been made against the lower end of the group and notice how the ogive has a reversed shape.

In each of Figures 7.7 and 7.8 we have added a fictitious group of zero frequency at one end of the distribution.

It is common practice to call the cumulative frequency graph, a cumulative frequency **polygon** if the points are joined by straight lines, and a cumulative frequency **curve** if the points are joined by a smooth curve.

(N.B. Unless you are told otherwise, always compile a "less than" cumulative frequency.)

All of these diagrams, of course, may be drawn from the original figures or on the basis of relative frequencies. In more advanced statistical work the latter are used almost exclusively and you should practise using relative frequencies whenever possible.

B. PICTOGRAMS

This is the simplest method of presenting information visually. These diagrams are variously called "pictograms", "ideograms", or "picturegrams" – the words all refer to the same thing. Their use is confined to the simplified presentation of statistical data for the general public.

Pictograms consist of simple pictures which represent quantities. There are two types and these are illustrated in the following examples. The data we will use is shown in Table 7.8.

Table 7.8: Cruises organised by a shipping line between Year 1 and Year 3

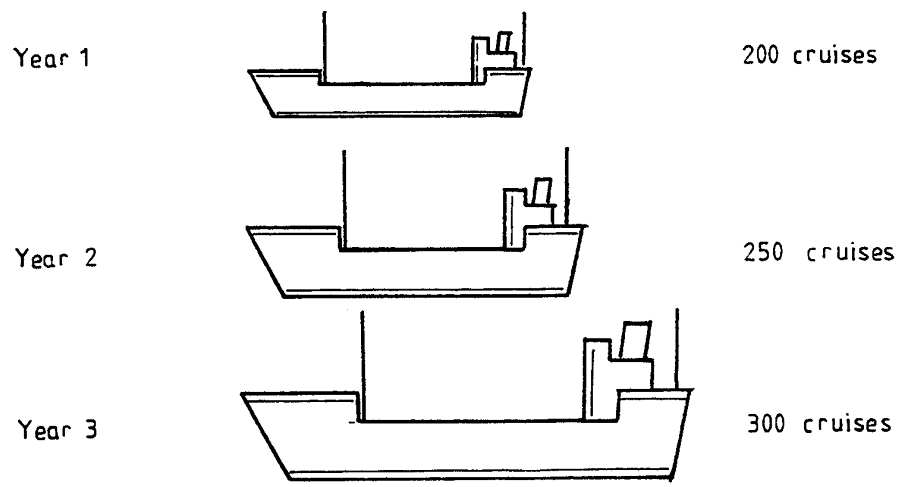
Year	Number of cruises	No. of passengers carried
1	200	100,000
2	250	140,000
3	300	180,000

Limited Form

We could represent the number of cruises by showing pictures of ships of varying size, as in Figure 7.9.

Figure 7.9: Number of cruises Years 1-3

(Source: Table 7.8)



Although these diagrams show that the number of cruises has increased each year, they can give false impressions of the actual increases. The reader can become confused as to whether the quantity is represented by the length or height of the pictograms, their area on the paper, or the volume of the object they represent. It is difficult to judge what increase has taken place. (Sometimes you will even find pictograms in which the sizes shown are actually *wrong* in relation to the real increases.)

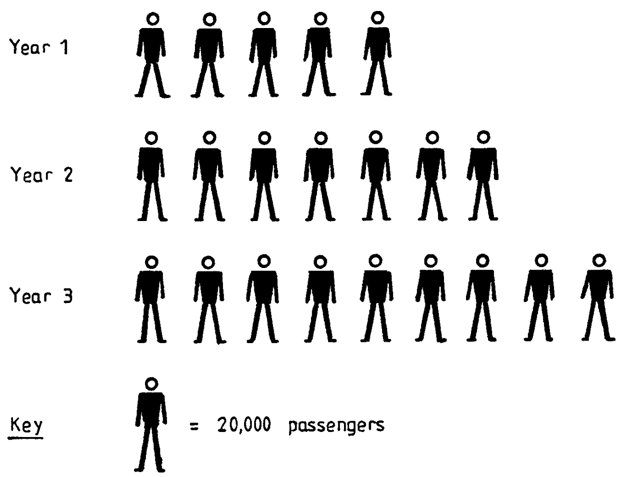
Accurate Form

If the purpose of the pictogram is to covey anything but the most minimal of information, to avoid confusion, it is better to try and give it a degree of accuracy in representing the numerical data. This is usually through representing specific quantities of the variable with pictures or graphics of some sort – usually related to the subject of the variable.

So, we could represent the numbers of passengers carried by using pictures of people as in Figure 7.10.

Figure 7.10: Number of passengers carried Years 1-3

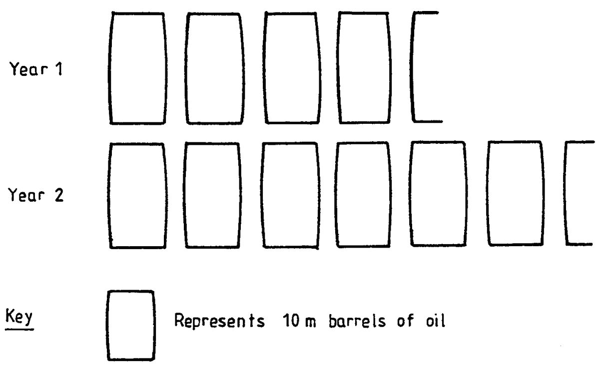
(Source: Table 7.8)



Each matchstick man is the same height and represents 20,000 passengers, so there can be no confusion over size.

These diagrams have no purpose other than generally presenting statistics in a simple way. They are not capable of providing real accurate detail. Consider Figure 7.11 –it is difficult to represent a quantity of less than 10m barrels – so we are left wondering what "[" means.

Figure 7.11: Imports of Crude Oil



C. PIE CHARTS

These diagrams, known also as circular diagrams, are used to show the manner in which various components add up to a total. Like pictograms, they are only used to display very simple information to non-expert readers. They are popular in computer graphics.

An example will show what the pie chart is. Suppose that we wish to illustrate the sales of gas in Great Britain in a certain year. The figures are taken from the Annual Abstract of Statistics as shown in Table 7.9.

Table 7.9: Gas sales in Great Britain in One Year

(Source: Annual Abstract of Statistics)

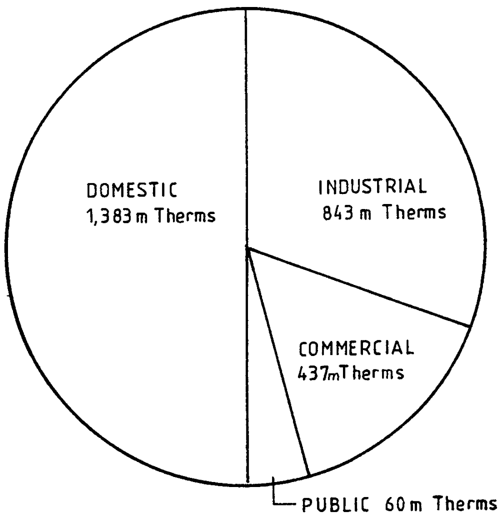
Uses	Million therms	%
Domestic	1,383	51
Industrial	843	31
Commercial	437	16
Public*	60	2
Total	2,723	100

* Central and local government uses, including public lighting

The figures are illustrated as the pie chart or circular diagram in Figure 7.12.

Figure 7.12: Example of a pie chart – gas sales in Great Britain

(Source: Table 7.9)



There are a number of simple steps to creating a pie chart.

- Tabulate the data and calculate the percentages.
- Convert the percentages into degrees, e.g.

$$51\% \text{ of } 360° = \frac{51}{100} \times 360° = 183.6°, \text{ etc.}$$

- Construct the diagram by means of a pair of compasses and a protractor. Don't overlook this point, because inaccurate and roughly drawn diagrams do not convey information effectively.
- Label the diagram clearly, using a separate "legend" or "key" if necessary. (A key is illustrated in Figure 7.13.) You can insert the exact figures for the each segment if more detail is required, since it is not possible to read this exactly from the diagram itself.
- If you have the choice, don't use a diagram of this kind with more than four or five component parts.

The main use of a pie chart is to show the relationship each component part bears to the whole. They are sometimes used side by side to provide comparisons, but this is not really to be recommended, unless the whole diagram in each case represents exactly the same total amount, as other diagrams (such as bar charts, which we discuss next) are much clearer.

D. BAR CHARTS

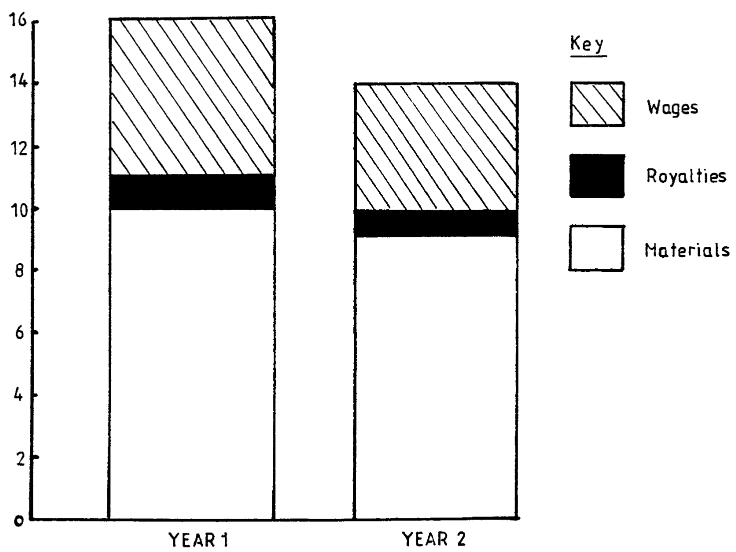
We have already met one kind of bar chart in the course of our studies of frequency distributions, namely the frequency bar chart. The bars there were simply thick lines representing, by their length, the frequencies of different values of the variate. The idea of a bar chart can, however, be extended beyond the field of frequency distributions, and we will now illustrate a number of the types of bar chart in common use.

We say "illustrate" because there are no rigid and fixed types, but only general ideas which are best studied by means of examples. You can supplement the examples in this study unit by looking at the commercial pages of newspapers and magazines.

Component Bar Chart

This first type of bar chart serves the same purpose as a circular diagram and, for that reason, is sometimes called a "component bar diagram" (see Figure 7.13).

Figure 7.13: Component bar chart showing cost of production of ZYX Co. Ltd



Note that the lengths of the components represent the amounts, and that the components are drawn in the same order so as to facilitate comparison. These bar charts are preferable to circular diagrams because:

- They are easily read, even when there are many components.
- They are more easily drawn.
- It is easier to compare several bars side by side than several circles.

Bar charts with vertical bars are sometimes called "column charts" to distinguish them from those in which the bars are horizontal (see Figure 7.14).

Figure 7.14: Horizontal bar chart of visitors arriving in the UK in one year

European	Commonwealth	USA	Others
51%	23%	22%	4%

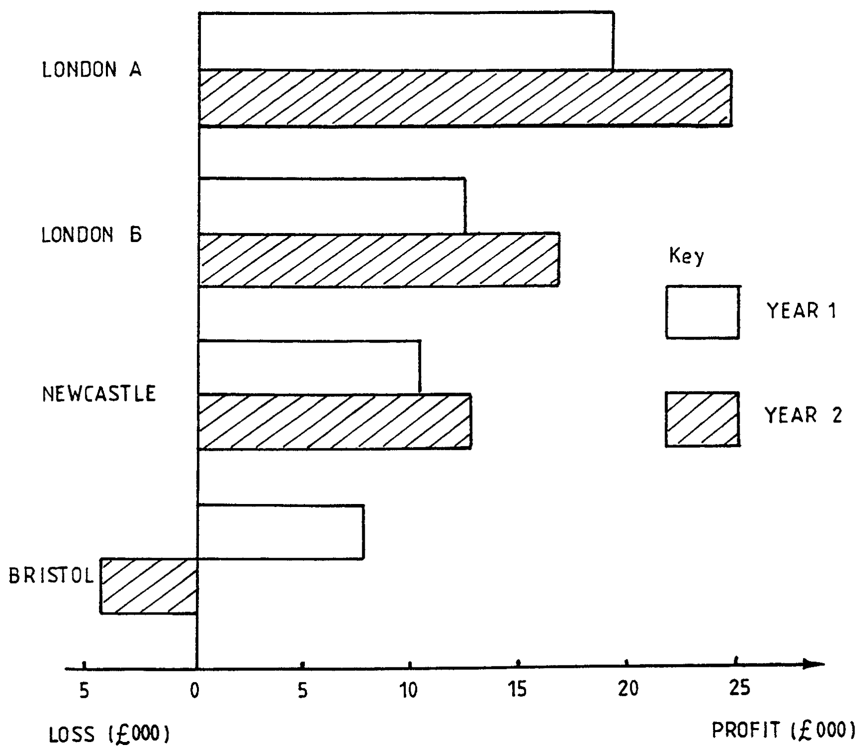
This figure is also an example of a percentage component bar chart, i.e. the information is expressed in percentages rather than in actual numbers of visitors.

If you compare several percentage component bar charts, you must be careful. Each bar chart will be the same length, as they each represent 100%, but they will not necessarily represent the same actual quantities – for example, 50% might have been 1 million, whereas in another year it may have been nearer to 4 million and in another to 8 million.

Horizontal Bar Chart

A typical case of presentation by a horizontal bar chart is shown in Figure 7.15. Note how a loss is shown by drawing the bar on the other side of the zero line.

Figure 7.15: Horizontal bar chart for the So and So Company Ltd to show profits made by branches in Year 1 and Year 2



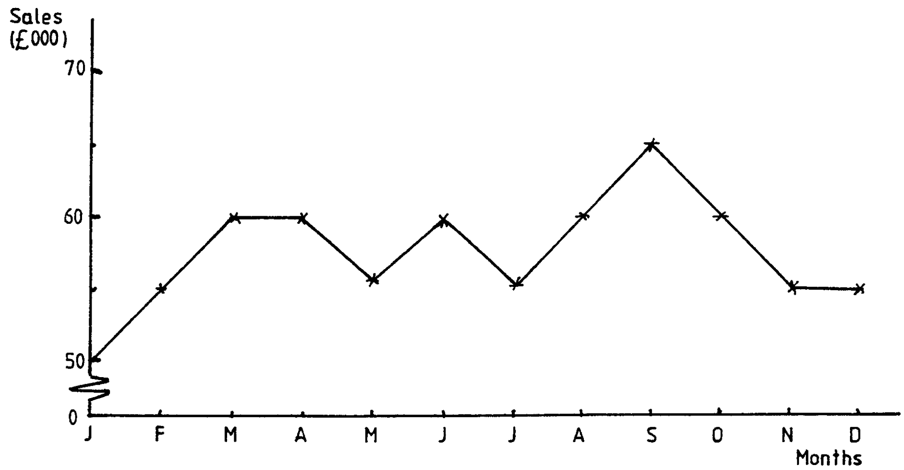
Pie charts and bar charts are especially useful for "***categorical***" variables as well as for numerical variables. The example in Figure 7.15 shows a categorical variable – i.e. the different branches form the different categories, whereas in Figure 7.13 we have a numerical variable, namely, time. Figure 7.15 is also an example of a multiple or compound bar chart as there is more than one bar for each category.

F. GENERAL RULES FOR GRAPHICAL PRESENTATION

There are a number of general rules which must be borne in mind when planning and using the graphical methods covered in this unit.

- Graphs and charts must be given clear but brief titles.
- The axes of graphs must be clearly labelled, and the scales of values clearly marked.
- Diagrams should be accompanied by the original data, or at least by a reference to the source of the data.
- Avoid excessive detail, as this defeats the object of diagrams.
- Wherever necessary, guidelines should be inserted to facilitate reading.
- Try to include the origins of scales. Obeying this rule sometimes leads to rather a waste of paper space. In such a case the graph could be "broken" as shown in Figure 7.16, but take care not to distort the graph by over-emphasising small variations.

Figure 7.16: Sales of LMN Factory



Practice Questions

These questions are typical of those likely to set in an examination.

1. The following figures show the number of faulty items made on successive days by a machine in an engineering workshop:

2	0	2	4	3	2	11	4	0	1
1	4	1	2	1	4	1	7	3	2
4	5	5	6	5	2	4	2	1	2
2	4	0	6	1	0	1	5	1	2
3	1	5	3	2	2	1	3	5	3
1	1	10	1	2	3	4	3	10	6
2	6	5	3	1	2	5	7	2	3
4	4	1	3	7	4	1	7	3	7
1	1	2	6	1	0	8	5	8	1
5	6	1	3	2	2	1	2	6	4

Tabulate these figures as a frequency distribution, giving columns containing the frequency, the relative frequency, the cumulative frequency and the relative cumulative frequency. Illustrate the distribution by means of a frequency bar chart of the relative figures.

2. Illustrate the following data by means of a histogram, a frequency polygon and an ogive:

Weekly Wage *(£)*	No. of Employees
31.01 to 36.00	6
36.01 to 41.00	8
41.01 to 46.00	12
46.01 to 51.00	18
51.01 to 56.00	25
56.01 to 61.00	30
61.01 to 66.00	24
66.01 to 71.00	14

3. Construct a circular diagram to illustrate the following figures:

Sources of Revenue in Ruritania for Year Ending 31 March

Source	Amount *(£000)*
Income Tax	1,863,400
Estate Duties	164,500
Customs & Excise	1,762,500
Miscellaneous	99,450

4. Illustrate by means of a bar chart the following data relating to the number of units sold of a particular kind of machine:

Year	Home Sales	Commonwealth Sales	Foreign Sales
1	6,500	5,000	5,800
2	6,250	5,400	5,700
3	5,850	6,600	5,500

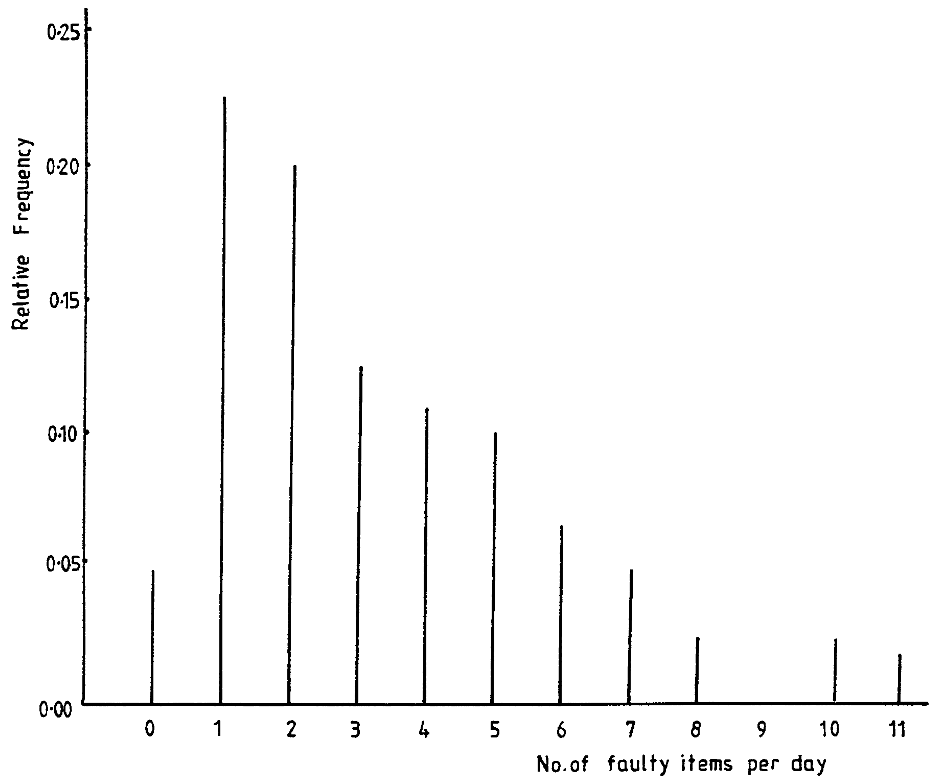
Now check your answers with the ones given at the end of the unit.

ANSWERS TO QUESTIONS FOR PRACTICE

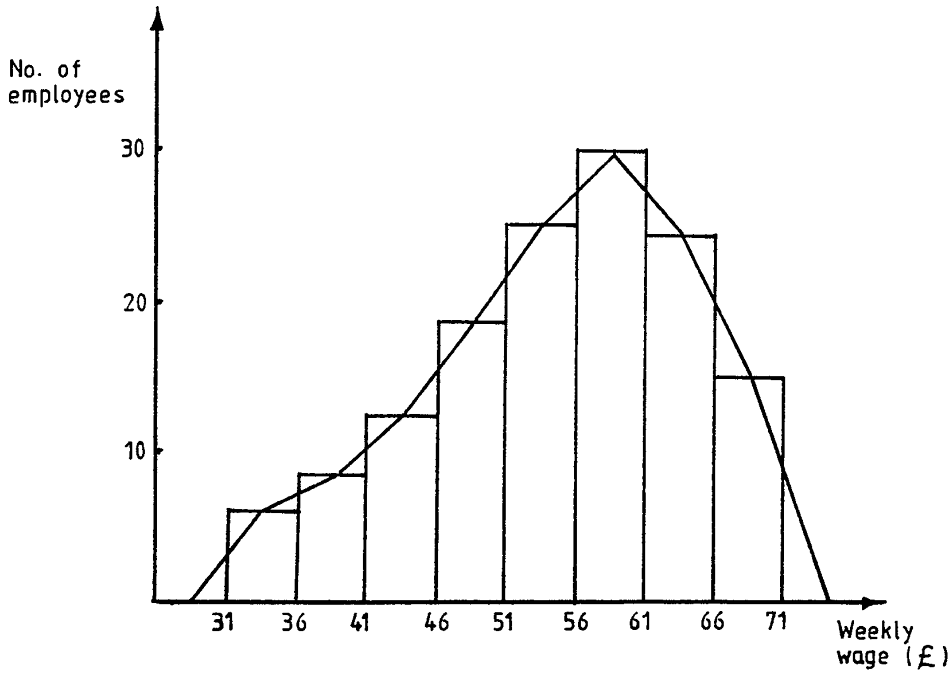
1.

Number of faulty items	Number of days with these items	Relative frequency	Cumulative frequency	Relative cumulative frequency
0	5	0.05	5	0.05
1	23	0.23	28	0.28
2	20	0.20	48	0.48
3	13	0.13	61	0.61
4	12	0.12	73	0.73
5	10	0.10	83	0.83
6	7	0.07	90	0.90
7	5	0.05	95	0.95
8	2	0.02	97	0.97
9	0	0.00	97	0.97
10	2	0.02	99	0.99
11	1	0.01	100	1.00
Total	100	1.00	–	–

Frequency bar chart – Faulty items produced by machine in workshop



2. ***Combined histogram and frequency polygon***

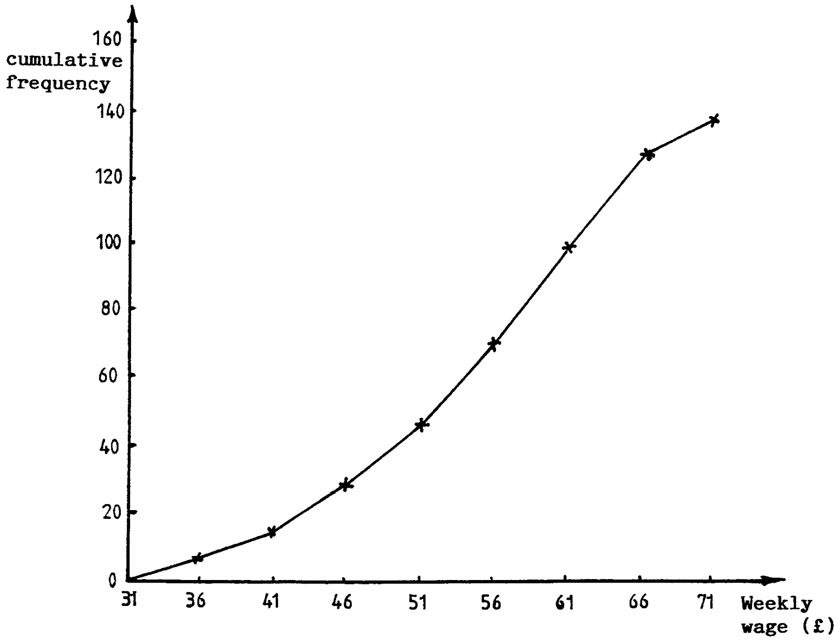


Workings for ogive:

Wage Group (£)	Frequency	Cumulative Frequency
31.01 – 36.00	6	6
36.01 – 41.00	8	14
41.01 – 46.00	12	26
46.01 – 51.00	18	44
51.01 – 56.00	25	69
56.01 – 61.00	30	99
61.01 – 66.00	24	123
66.01 – 71.00	14	137

The ogive follows:

Ogive for wages distribution

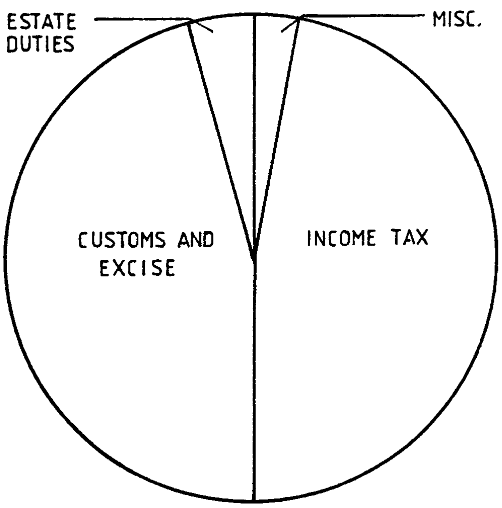


3. The amounts must first be turned into percentages and degrees.

Income tax	47.9%	172°
Estate duties	4.2%	15°
Customs & excise	45.3%	163°
Miscellaneous	2.6%	9°
Total	100.0%	359°

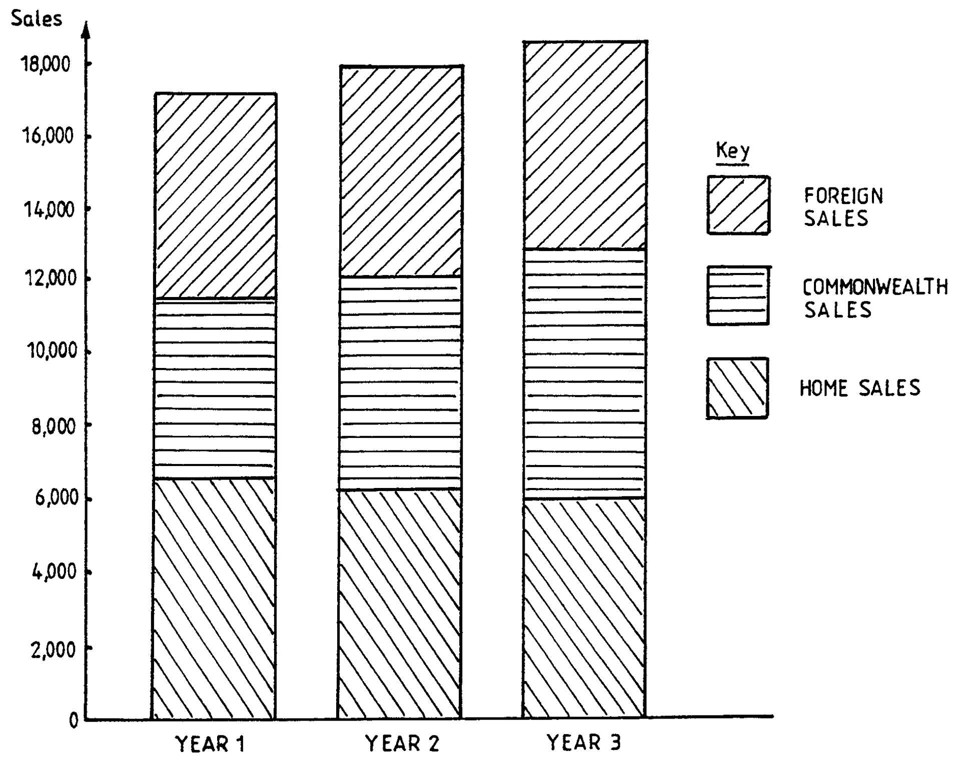
Note: The angles do not add up to 360° exactly because of rounding errors.

Figure 30: Sources of Revenue in Ruritania



4. The bar chart is shown in below. Shading has been added for clarity. You can, of course, draw bar charts on a percentage basis.

Bar Chart of Machine Sales



Study Unit 8

Measures of Location

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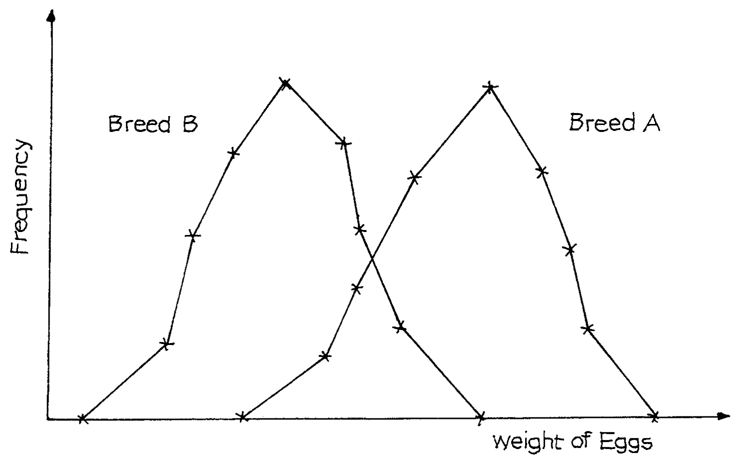
INTRODUCTION

We have looked at frequency distributions in detail in a previous study unit and you should, by means of a quick revision, make sure that you have understood them before proceeding.

A frequency distribution may be used to give us concise information about its variate, but often, we wish to compare two or more distributions. Consider, for example, the distribution of the weights of eggs from two different breeds of poultry (which is a topic in which you would be interested if you worked for an egg marketing company). Having weighed a large number of eggs from each breed, we might have compiled frequency distributions and graphed the results. The two frequency polygons might well look something like Figure 8.1.

Figure 8.1: Frequency polygon showing weight of eggs from two breeds of hen



Examining these distributions you will see that they look alike except for one thing – they are located on different parts of the scale. In this case the distributions overlap and, although some eggs from Breed A are of less weight than some eggs from Breed B, eggs from Breed A are, in general, heavier than those from Breed B.

Remember that one of the objects of statistical analysis is to condense unwieldy data so as to make it more readily understood. The drawing of frequency curves has enabled us to make an important general statement concerning the relative egg weights of the two breeds of poultry, but we would now like to take the matter further and calculate some figure which will serve to indicate the general level of the variable under discussion.

In everyday life, we commonly use such a figure when we talk about the "average" value of something or other. We might have said, in reference to the two kinds of egg, that those from Breed A had a higher average weight than those from Breed B. Distributions with different averages indicate that there is a different general level of the variate in the two groups.

The single value which we use to describe the general level of the variate is called a "***measure of location***" or a "***measure of central tendency***". There are three such measures with which you need to be familiar:

- The arithmetic mean
- The mode
- The median.

A. THE ARITHMETIC MEAN

This is what we normally think of as the "average" of a set of values. It is obtained by adding together all the values and then dividing the total by the number of values involved.

Take, for example, the following set of values which are the heights, in inches, of seven men:

Table 8.1: Height of seven men

Man	Height *(ins)*
A	74
B	63
C	64
D	71
E	71
F	66
G	74
Total	483

The arithmetic mean of these heights is 483 ÷ 7 = 69 ins. Notice that some values occur more than once, but we still add them all.

At this point we must introduce a little algebra. We don't always want to specify what particular items we are discussing (heights, egg weights, wages, etc.) and so, for general discussion, we use, as you will recall from algebra, some general letter, usually x. Also, we indicate the sum of a number of x's by Σ (***sigma***).

Thus, in our example, we may write:

$$\Sigma x = 483$$

We shall be using "***sigma notation***" extensively in this unit, so make sure you understand the meaning of the expression.

We indicate the arithmetic mean by the symbol $\bar{x}$ (called "***x bar***") and the number of items by the letter n. The calculation of the arithmetic mean can be described by formula thus:

$$\bar{x} = \frac{\Sigma x}{n}$$

or $$\bar{x} = \frac{1}{n}\Sigma x$$

The latter expression is customary in statistical work. Applying it to the example above, we have:

$$\bar{x} = \frac{1}{7}(483) = 69 \text{ ins}$$

You will often find the arithmetic mean simply referred to as "the mean" when there is no chance of confusion with other means (which we are not concerned with here).

The Mean of a Simple Frequency Distribution

When there are many items (i.e. when n is large), the arithmetic can be eased somewhat by forming a frequency distribution. Consider the following example.

Example 1

Table 8.2: Height distribution data

Height in inches *(x)*	No. of men at this height *(f)*	Product *(fx)*
63	1	63
64	1	64
66	1	66
71	2	142
74	2	148
Total	$\Sigma f = 7$	$\Sigma(fx) = 483$

Indicating the frequency of each value by the letter f, you can see that $\Sigma f = n$ and that, when the x's are not all the separate values but only the different ones, the formula becomes:

$$\bar{x} = \frac{\Sigma(fx)}{\Sigma f}$$

which is 483/7 = 69 ins as before.

Of course, with only seven items it would not be necessary, in practice, to use this method, but if we had a much larger number of items the method would save a lot of additions.

Example 2

Consider the following table and, as practice, complete the (fx) column and calculate the value of the arithmetic mean, $\bar{x}$. Do this before moving on.

Table 8.3: Data for calculation of arithmetic mean

Value of x	Number of items *(f)*	Product *(fx)*
3	1	3
4	4	16
5	7	35
6	15	
7	35	
8	24	
9	8	
10	6	
Total		

You should have obtained the following answers:

The total numbers of items, $\Sigma f = 100$

The total product, $\Sigma(fx) = 713$

The arithmetic mean, $\bar{x} = \frac{713}{100} = 7.13$

Make sure that you understand this study unit so far. Revise it if necessary, before going on to the next paragraph. It is most important that you do not get muddled about calculating arithmetic means.

The Mean of a Grouped Frequency Distribution

Suppose now that you have a grouped frequency distribution. In this case, you will remember, we do not know the actual individual values, only the groups in which they lie. How, then, can we calculate the arithmetic mean? The answer is that we cannot calculate the exact value of $\bar{x}$, but we can make an approximation sufficiently accurate for most statistical purposes. We do this by assuming that all the values in any group are equal to the mid-point of that group.

The procedure is very similar to that for a simple frequency distribution (which is why we stressed the need for revision) and is shown in this example:

Table 8.4: Data for calculation of arithmetic mean of a grouped frequency distribution

Group	Mid-value *(x)*	Frequency *(f)*	Product *(fx)*
0 – 10	5	3	15
10 – 20	15	6	90
20 – 30	25	10	250
30 – 40	35	16	560
40 – 50	45	9	405
50 – 60	55	5	275
60 – 70	65	1	65
Total		$\Sigma f = 50$	$\Sigma(fx) = 1,660$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{1}{50} \times 1,660 = 33.2$$

Provided that Σf is not less than about 50 and that the number of groups is not less than about 12, the arithmetic mean thus calculated is sufficiently accurate for all practical purposes.

There is, though, one pitfall to be avoided when using this method – if all the groups should not have the same class interval, be sure that you get the correct mid-values! The following is part of a table with varying class intervals, to illustrate the point:

Table 8.5: Data for grouped frequency distribution

Group	Mid-value *(x)*
0 – 10	5
10 – 20	15
20 – 40	30
40 – 60	50
60 – 100	80

You will remember that in discussing the drawing of histograms we had to deal with the case where the last group was not exactly specified. The same rules for drawing the histogram apply to the calculation of the arithmetic mean.

Simplified Calculation

It is possible to simplify the arithmetic involved in calculating an arithmetic mean still further by the following two devices:

- Working from an assumed mean in the middle of one convenient class.
- Working in class intervals instead of in the original units.

Let us consider the first device.

(a) Working from an assumed mean in the middle of one convenient class

If you go back to our earlier examples you will discover after some arithmetic that if you add up the differences in value between each reading and the true mean, then these differences add up to zero.

Take first the height distribution considered at the start of this section:

Table 8.6: Height distribution data

$\bar{x} = 69$ ins

Man	Height, x, *(ins)*	$(x - \bar{x})$ *(ins)*
A	74	5
B	63	–6
C	64	–5
D	71	2
E	71	2
F	66	–3
G	74	5
	$\Sigma(x - \bar{x}) = +14 - 14 = 0$	

i.e. $\Sigma(x - \bar{x}) = 0$

Secondly, consider the grouped frequency distribution given earlier in this section:

Table 8.7: Data for grouped frequency distribution

$\bar{x} = 33.2$

Group	Mid-value *(x)*	Frequency *(f)*	$(x - \bar{x})$	$f(x - \bar{x})$
0 – 10	5	3	–28.2	–84.6
10 – 20	15	6	–18.2	–109.2
20 – 30	25	10	–8.2	–82
30 – 40	35	16	1.8	28.8
40 – 50	45	9	11.8	106.2
50 – 60	55	5	21.8	109
60 – 70	65	1	31.8	31.8
Total		$\Sigma f = 50$	$\Sigma(f(x - \bar{x})) = +275.8 - 275.8 = 0$	

i.e. $\Sigma f(x - \bar{x}) = 0$

If we take any value other than $\bar{x}$ and follow the same procedure, the sum of the differences (sometimes called deviations) will not be zero. In our first example, let us assume the mean to be 68 ins and label the assumed mean x_o. The differences between each reading and this assumed value are:

Table 8.8: Differences between x and x_o for height distribution data

Man	Height, x, *(ins)*	$d = (x - x_o)$ *(ins)*
A	74	6
B	63	–5
C	64	–4
D	71	3
E	71	3
F	66	–2
G	74	6
	$\Sigma(x - x_o) = +18 - 11 = 7$	

i.e. $\Sigma(x - x_o) = +7$ ins or $\Sigma d = +7$ ins

We can make use of this property and use it as a "short-cut" for finding $\bar{x}$.

Firstly, we have to choose some value of x as an ***assumed mean***. We try to choose it near to where we think the true mean, $\bar{x}$, will lie, and we always choose it as the mid-point of one of the groups when we are involved with a grouped frequency distribution.

In the above example, the total deviation, d, does not equal zero, so 68 cannot be the true mean. As the total deviation is positive, we must have *underestimated* in our choice of x_o, so the true mean is higher than 68. As there are seven readings, we need to adjust x_o upwards by one seventh of the total deviation, i.e. by (+7) ÷ 7 = +1. Therefore the true value of $\bar{x}$ is:

$$68 + \frac{(+7)}{7} = 68 + 1 = 69 \text{ ins}$$

We know this to be the correct answer from our earlier work.

Let us now illustrate the "short-cut" method for the grouped frequency distribution. We shall take x_o as 35 as this is the mid-value in the centre of the distribution.

Table 8.9: Differences between x and x_o for grouped frequency distribution data

Group	Mid-value *(x)*	Frequency *(f)*	$d = (x - x_o)$	$f(x - x_o) = fd$
0 – 10	5	3	–30	–90
10 – 20	15	6	–20	–120
20 – 30	25	10	–10	–100
30 – 40	35	16	0	0
40 – 50	45	9	10	90
50 – 60	55	5	20	100
60 – 70	65	1	30	30
Total		Σf = 50	Σfd = –310 + 220 = –90	

$$\Sigma fd = -90$$

This time we must have *overestimated* x_o, as the total deviation, Σfd, is negative. As there are 50 readings altogether, the true mean must be $\frac{1}{50}$th of (–90) lower than 35, i.e.

$$\bar{x} = 35 + \frac{(-90)}{50} = 35 - 1.8 = 33.2$$

which is as we found previously.

(b) Working in class intervals instead of in the original units.

This method of calculation can be used with a grouped frequency distribution to work in units of the class interval instead of in the original units. In the fourth column of Table 8.9, you can see that all the deviations are multiples of 10, so we could have worked in units of 10 throughout and then compensated for this at the end of the calculation.

Let us repeat the calculation using this method. The result (with $x_o = 35$) is set out in Table 8.10.

Note that, here, the symbol used for the length of the class interval is c, although you may also come across the symbol i used for this purpose.

Table 8.10: Differences between x and x_o for grouped frequency distribution data (using class intervals)

Group	Mid-value *(x)*	Frequency *(f)*	$(x - x_o)$	$d = \frac{x - x_o}{c}$	fd
0 – 10	5	3	–30	–3	–9
10 – 20	15	6	–20	–2	–12
20 – 30	25	10	–10	–1	–10
30 – 40	35	16	0	0	0
40 – 50	45	9	10	1	9
50 – 60	55	5	20	2	10
60 – 70	65	1	30	3	3
Total		$\Sigma f = 50$	$\Sigma \frac{x - x_o}{c} = +22 - 31 = -9$		

Remembering that we need to compensate, by multiplying by 10, for working in class interval units in the table:

$$\bar{x} = 35 + \frac{(-9)}{50} \times 10$$

$$\bar{x} = 35 - \frac{90}{50} = 35 - 1.8 = 33.2 \text{ as before.}$$

The general formula for $\bar{x}$ that applies for all grouped frequency distributions having equal class intervals can be written as:

$$\bar{x} = x_o + \frac{\Sigma fd}{n} \times c$$

where $d = \frac{x - x_o}{c}$

As we mentioned at an earlier stage, you have to be very careful if the class intervals are unequal, because you can only use one such interval as your working unit. Table 8.11 shows how to deal with this situation.

Table 8.11: Differences between x and x_o for grouped frequency distribution data (using unequal class intervals)

Group	Mid-value *(x)*	$d = \frac{x - x_o}{c}$	Frequency *(f)*	Product *(fd)*
0 – 10	5	–3	3	–9
10 – 20	15	–2	6	–12
20 – 30	25	–1	10	–10
30 – 40	35	0	16	0
40 – 50	45	+1	9	+9
50 – 70	60	+2½	6	+15
Total			50	+24 + (–31) = –7

The assumed mean is 35, as before, and the working unit is a class interval of 10. Notice how d for the last group is worked out: the mid-point is 60, which is 2½ times 10 above the assumed mean. The required arithmetic mean is, therefore:

$$\bar{x} = 35 - \frac{7 \times 10}{50} = 35 - \frac{70}{50} = 35 - 1.4 = 33.6$$

We have reached a slightly different figure from before because of the error introduced by the coarser grouping in the “50 – 70” region.

The method just described is of great importance. By using it correctly, you can often do the calculations for very complicated-looking distributions by using mental arithmetic and pencil and paper. With the advent of electronic calculators, the time saving on calculations of the arithmetic mean is not great, but the method is still preferable because:

- The number involved are smaller and thus you are less likely to make slips in arithmetic.
- The method can be extended to enable us to find easily the standard deviation of a frequency distribution (which we shall examine in the next unit).

Characteristics of the Arithmetic Mean

There are a number of characteristics of the arithmetic mean which you must know and understand.

- It is not necessary to know the value of every item in order to calculate the arithmetic mean. Only the total and the number of items are needed. For example, if you know the total wages bill and the number of employees, you can calculate the arithmetic mean wage without knowing the wages of each person.
- It is fully representative because it is based on all, and not just some, of the items in the distribution.
- One or two extreme values can make the arithmetic mean somewhat unreal by their influence on it. For example, if a millionaire came to live in a country village, the inclusion of his/her income in the arithmetic mean for the village might make the place seem very much better off than it really was!
- The arithmetic mean is reasonably easy to calculate and to understand.
- In more advanced statistical work it has the advantage of being amenable to algebraic manipulation.

B. THE MODE

The first alternative to the mean which we will discuss is the mode. This is the name given to ***the most frequently occurring value***.

Mode of a Simple Frequency Distribution

Consider the following frequency distribution:

Table 8.12: Accident distribution data

Number of accidents per day *(x)*	Number of days with the stated number of accidents *(f)*
0	27
1	39
2	30
3	20
4	7
Total	123

In this case the most frequently occurring value is 1 (it occurred 39 times) and so the mode of this distribution is 1.

Note that the mode, like the mean, is a value of the variate, x, not the frequency of that value. A common error is to say that the mode of the above distribution is 39. *This is wrong*. The mode is 1. Watch out, and do not fall into this trap!

For comparison, calculate the arithmetic mean of the distribution – it works out at 1.52.

The mode is used in those cases where it is essential for the measure of location to be an actually occurring value. An example is the case of a survey carried out by a clothing store to determine what size of garment to stock in the greatest quantity. The average size of garment in demand might turn out to be, let us say, 9.3724, which is not an actually occurring value and doesn't help us to answer our problem. However, the mode of the distribution obtained from the survey would be an actual value (perhaps size 8) and it would provide the answer to the problem.

Mode of a Grouped Frequency Distribution

When the data is given in the form of a grouped frequency distribution, it is not quite so easy to determine the mode. What, you might ask, is the mode of the following distribution?

Table 8.13: Data for determining the mode of a grouped frequency distribution

Group	Frequency
0 – 10	4
10 – 20	6
20 – 30	10
30 – 40	16
40 – 50	24
50 – 60	32
60 – 70	38
70 – 80	40
80 – 90	20
90 – 100	10
100 – 110	5
110 – 120	1

All we can really say is that "70 – 80" is the ***modal group*** (the group with the largest frequency).

You may be tempted to say that the mode is 75, but this is not true, nor even a useful approximation in most cases. The reason is that the modal group depends on the method of grouping, which can be chosen quite arbitrarily to suit our convenience. The distribution could have been set out with class intervals of five instead of 10, and would then have appeared as follows (only the middle part is shown, to illustrate the point):

Table 8.14: Revised data for determining the mode of a grouped frequency distribution

Group	Frequency
60 – 65	16
65 – 70	22
70 – 75	21
75 – 80	19
80 – 85	12
85 – 90	8

The modal group is now "65 – 70". Likewise, we will get different modal groups if the grouping is by 15 or by 20 or by any other class interval, and so the mid-point of the modal group is not a good way of estimating the mode.

In practical work, this determination of the modal group is usually sufficient, but you may occasionally be asked for the mode to be determined from a grouped distribution.

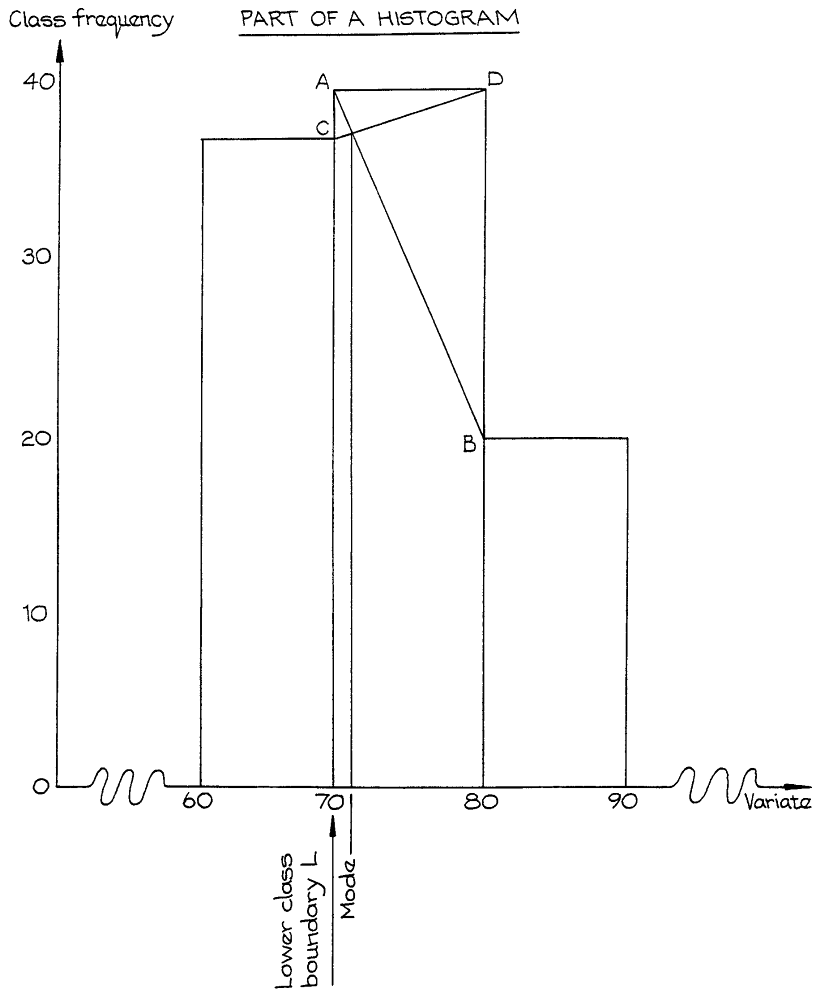
A number of procedures based on the frequencies in the groups adjacent to the modal group can be used, and we shall now describe one procedure. You should note, however, that these procedures are only mathematical devices for finding the ***most likely*** position of the mode – it is not possible to calculate an exact and true value in a grouped distribution.

We saw that the modal group of our original distribution was "70 – 80". Now examine the groups on each side of the modal group; the group below (i.e. 60 – 70) has a frequency of 38, and the one above (i.e. 80 – 90) has a frequency of 20. This suggests to us that the mode may be some way towards the lower end of the modal group rather than at the centre. A graphical method for estimating the mode is shown in Figure 8.2.

This method can be used when the distribution has equal class intervals. Draw that part of the histogram which covers the modal class and the adjacent classes on either side.

Draw in the diagonals AB and CD as shown in Figure 8.2. From the point of intersection draw a vertical line downwards. Where this line crosses the horizontal axis is the mode. In our example the mode is just less than 71.

Figure 8.2: Using a histogram to determine the most likely value of the mode



Characteristics of the Mode

Some of the characteristics of the mode are worth noting, particularly as they compare with those of the arithmetic mean.

- The mode is very easy to find with ungrouped distributions, since no calculation is required.
- It can only be determined roughly with grouped distributions.
- It is not affected by the occurrence of extreme values.

- Unlike the arithmetic mean, it is not based on all the items in the distribution, but only on those near its value.
- In ungrouped distributions the mode is an actually occurring value.
- It is not amenable to the algebraic manipulation needed in advanced statistical work.
- It is not unique, i.e. there can be more than one mode – for example, in the set of numbers, 6, 7, 7, 7, 8, 8, 9, 10, 10, 10, 12, 13, there are two modes, namely 7 and 10. This set of numbers would be referred to as having a **bimodal** distribution.
- The mode may not exist. For example, in the set of numbers 7, 8, 10, 11, 12, each number occurs only once so this distribution has no mode.

C. THE MEDIAN

The desirable feature of any measure of location is that it should be near the middle of the distribution to which it refers. Now, if a value is near the middle of the distribution, then we expect about half of the distribution to have larger values, and the other half to have smaller values. This suggests to us that a possible measure of location might be that value which is such that exactly half (i.e. 50%) of the distribution has larger values and exactly half has lower values. The value which so divides the distribution into equal parts is called the ***median***.

Look at the following set of values:

6, 7, 7, 8, 8, 9, 10, 10, 10, 12, 13

The total of these eleven numbers is 100 and the arithmetic mean is therefore 100/11 = 9.091, while the mode is 10 because that is the number which occurs most often (three times). The median, however, is 9 because there are five values above and five values below 9.

Our first rule for determining the median is, therefore, as follows:

- Arrange all the values in order of magnitude and the median is then the middle value.

Note that *all* the values are to be used – even though some of them may be repeated, they must all be put separately into the list. In the example just dealt with, it was easy to pick out the middle value because there was an odd number of values. But what if there is an even number? Then, by convention, the median is taken to be the arithmetic mean of the two values in the middle. For example, take the following set of values:

6, 7, 7, 8, 8, 9, 10, 10, 11, 12

The two values in the middle are 8 and 9, so the median is 8½.

Median of a Simple Frequency Distribution

Statistical data, of course, is rarely in such small groups and, as you have already learned, we usually deal with frequency distributions. How, then, do we find the median if our data is in the form of a distribution?

Let us take the example of the frequency distribution of accidents already used in discussing the mode. The total number of values is 123 and so when those values are arranged in order of magnitude, the median will be the 62nd item because that will be the middle item. To see what the value of the 62nd item will be, let us draw up the distribution again. Table 8.15 repeats the data from Table 8.12, but adds a column setting out the cumulative frequency.

Table 8.15: Accident distribution data

Number of accidents per day *(x)*	Number of days with the stated number of accidents *(f)*	Cumulative frequency *(F)*
0	27	27
1	39	66
2	30	96
3	20	116
4	7	123
Total	123	

You can see from the last column that, if we were to list all the separate values in order, the first 27 would all be 0s and from then up to the 66th would be 1s. It follows, therefore, that the 62nd item would be a 1 and that the median of this distribution is 1.

Median of a Grouped Frequency Distribution

The final problem connected with the median is how to find it when our data is in the form of a grouped distribution. The solution to the problem, as you might expect, is very similar to the solution for an ungrouped distribution – we halve the total frequency and then find, from the cumulative frequency column, the corresponding value of the variate.

Because a grouped frequency distribution nearly always has a large total frequency, and because we do not know the exact values of the items in each group, it is not necessary to find the **two** middle items when the total frequency is even – just halve the total frequency and use the answer (whether it is a whole number or not) for the subsequent calculation.

Table 8.16: Data for determining the median of a grouped frequency distribution

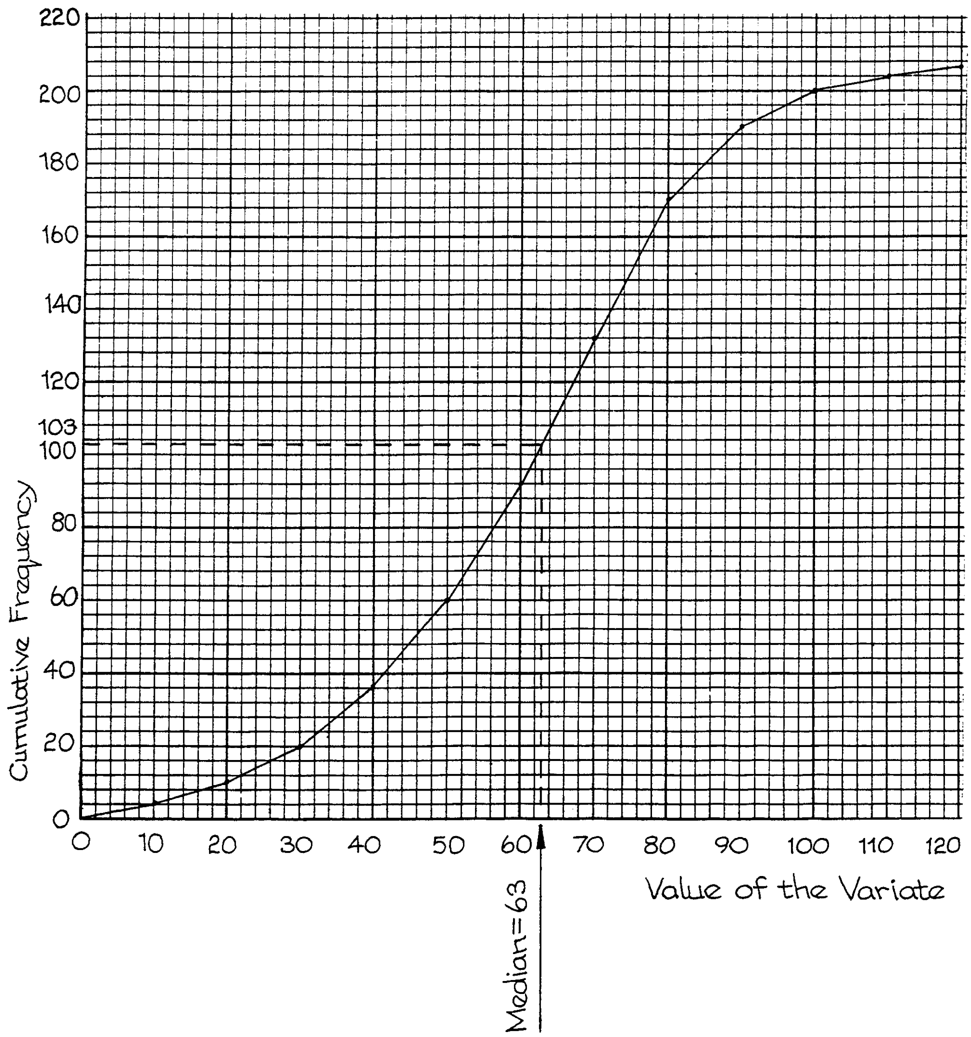
Group	Frequency *(f)*	Cumulative frequency *(F)*
0 – 10	4	4
10 – 20	6	10
20 – 30	10	20
30 – 40	16	36
40 – 50	24	60
50 – 60	32	92
60 – 70	38	130
70 – 80	40	170
80 – 90	20	190
90 – 100	10	200
100 – 110	5	205
110 – 120	1	206
Total	206	

The total frequency is 206 and therefore the median is the 103rd item which, from the cumulative frequency column, must lie in the 60 – 70 group. But exactly where in the 60-70 group? Well, there are 92 items before we get to that group and we need the 103rd item, so we obviously need to move into that group by 11 items. Altogether in our 60 – 70 group there are 38 items, so we need to move 11/38 of the way into that group, that is 11/38 of 10 above 60. Our median is therefore:

$$60 + \frac{110}{38} = 60 + 2.89 = 62.89$$

The use of the cumulative frequency distribution will, no doubt, remind you of its graphical representation, the ogive. In practice, a convenient way to find the median of a grouped distribution is to draw the ogive and then, against a cumulative frequency of half the total frequency, to read off the median. In our example the median would be read against 103 on the cumulative frequency scale (see Figure 8.3). If the ogive is drawn with relative frequencies, then the median is always read off against 50%.

Figure 8.3: Reading the median from an ogive



Characteristics of the Median

Characteristic features of the median, which you should compare with those of the mean and the mode, are as follows:

- It is fairly easily obtained in most cases, and is readily understood as being the "half-way point".
- It is less affected by extreme values than the mean. The millionaire in the country village might alter considerably the mean income of the village but he would have almost no effect at all on the median.
- It can be obtained without actually having all the values. If, for example, we want to know the median height of a group of 21 men, we do not have to measure the height of every single one; it is only necessary to stand the men in order of their heights and then only the middle one (No. 11) need be measured, for his height will be the median height. The median is thus of value when we have open-ended classes at the edges of the distribution as its calculation does not depend on the precise values of the variate in these classes, whereas the value of the arithmetic mean does.
- The median is not very amenable to further algebraic manipulation.

Practice Questions

1. The following table shows the consumption of electricity of 100 householders during a particular week. Calculate the arithmetic mean consumption of the 100 householders.

Consumption of electricity of 100 householders (one week)

Consumption *(kilowatt hours)*	Number of householders
0 – under 10	5
10 – ″ 20	8
20 – ″ 30	20
30 – ″ 40	29
40 – ″ 50	20
50 – ″ 60	11
60 – ″ 70	6
70 – ″ 80	1
80 or over	0

2. If you were working for a local Health Department and were studying the physique of men in a particular village, you might, as part of the job, measure the heights of men in a sample of a dozen or so from the village. If, after doing this, you find that your sample contains two brothers who are retired circus dwarfs, what sort of "average" would you calculate from the sample, and why?

3. The rate of effluent discharge from two plants based on different systems was sampled at four-hourly intervals and the results are given below:

System A

10,	12,	10,	11,	9,	12,	14,	8,	9,	12,	7,	13,	10,	12,	11,
13,	8,	10,	9,	12,	13,	10,	16,	12,	7,	6,	5,	10,	11,	14.

System B

10,	17,	18,	6,	5,	10,	11,	13,	6,	15,	14,	10,	12,	9,	8,
7,	4,	18,	19,	10,	12,	13,	16,	9,	12,	3,	4,	12,	4,	9.

Calculate the mean rate of discharge for each system.

4. Using the "short-cut" methods given in this study unit, calculate the arithmetic mean (i.e. the average) size of shareholding, from the following grouped distribution:

Distribution of shareholding

Number of shares	Number of persons with this size of holding
Less than 50	23
50 – 99	55
100 – 199	122
200 – 299	136
300 – 399	100
400 – 499	95
500 – 599	81
600 – 699	63
700 – 799	17
800 – 1,000	8

5. A garage keeps two cars for daily hire. The number of cars asked for in a day has varied as follows:

Cars wanted in a day	Proportion of days
0	0.36
1	0.38
2	0.18
3	0.06
4	0.02
5 or more	0

(a) What is the mean daily demand?

(b) What is the mean number supplied (i.e. hired out) per day?

(c) On what fraction of days is some demand turned away?

6. The following distribution gives the weight in lbs of 100 schoolboys. Find the median weight.

Weight in lbs	No. of schoolboys
100 – 109	2
110 – 119	15
120 – 129	37
130 – 139	31
140 – 149	11
150 – 159	4
	100

7. The holdings of shares (nominal value £1) in DEF Plc are given in the following table. Calculate the arithmetic mean holding, the median holding and the modal holding.

Number of shares held	No. of persons with these shares	Number of shares held	No. of persons with these shares
Under50	80	300 – 349	28
50 – 99	142	350 – 399	15
100 – 149	110	400 –449	9
150 – 199	75	450 – 499	7
200 – 249	52	500 – 549	3
250 – 299	38	550 – 599	2

Now check your answers with the ones given at the end of the unit.

ANSWERS TO QUESTIONS FOR PRACTICE

1.

Consumption of electricity of 100 householders (one week)

Consumption *(kwH)*	Mid-value *(x)*	Number of householders *(f)*	fx
0 – under 10	5	5	25
10 – ″ 20	15	8	120
20 – ″ 30	25	20	500
30 – ″ 40	35	29	1,015
40 – ″ 50	45	20	900
50 – ″ 60	55	11	605
60 – ″ 70	65	6	390
70 – ″ 80	75	1	75
80 or over		0	0
		100	3,630

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{3{,}630}{100} = 36.3$$

i.e. the arithmetic mean of the householders' consumption is 36.3 kilowatt hours.

2. In order to give the correct impression of the general stature of the mean in the village, it would be best to use the median. Normally one thinks of the arithmetic mean as the best measure of "average", but in this case it would be too much affected by the two extremely low values to be representative. The mode would not be so affected, but with such a small sample (a dozen or so) it would be erratic, or might not exist if all the heights were different.

3. You can group the data first if you wish, but it is probably quicker just to add the numbers as they stand.

System A: Total number of readings = 30

Total of rates of discharge = 316

Mean rate of discharge $= \frac{316}{30} = 10.53$

System B: Total number of readings = 30

Total of rates of discharge = 316

Mean rate of discharge $= \frac{316}{30} = 10.53$

4. Take c = 100, the most common class interval, and x_o = 250, the middle of the modal group.

Distribution of shareholding

No. of shares	Mid-Value *(x)*	No. of persons with a holding of x *(f)*	$d = \frac{x - x_o}{c}$	fd
Less than 50	25	23	–2¼	–51¾
50 – 99	75	55	–1¾	–96¼
100 – 199	150	122	–1	–122
200 – 299	250	136	0	0
300 – 399	350	100	+1	+100
400 – 499	450	95	+2	+190
500 – 599	550	81	+3	+243
600 – 699	650	63	+4	+252
700 – 799	750	17	+5	+85
800 – 1,000	900	8	+6½	+52
Total		700		–270 +922 = +652

$$\bar{x} = 250 + \frac{652}{700} \times 100$$

$$= 250 + 93.1 = 343 \text{ to the nearest whole number}$$

Therefore, the mean shareholding = 343 shares.

5. (a)

Demand *(x_1)*	Frequency (as a proportion) *(f)*	fx_1
0	0.36	0
1	0.38	0.38
2	0.18	0.36
3	0.06	0.18
4	0.02	0.08
5 or more	0	0
	1.00	1.00

$$\text{Mean demand} = \frac{\Sigma fx_1}{\Sigma f} = \frac{1.00}{1.00} = 1 \text{ car per day}$$

(b)

Supply (x_2)	Frequency (as a proportion) (f)	fx_2
0	0.36	0
1	0.38	0.38
2	(0.18 + 0.06 + 0.02 =) 0.26	0.52
	1.00	0.90

$$\text{Mean supply} = \frac{\Sigma fx_2}{\Sigma f} = \frac{0.90}{1.00} = 0.90 \text{ cars per day}$$

(c) Demand is turned away whenever three or more cars are requested in one day. This proportion of days is 0.06 + 0.02 = 0.08. Therefore, on eight days in every 100 demand is turned away.

6.

Weight in lbs	No. of schoolboys	Cumulative Frequency
100 – 109	2	2
110 – 119	15	17
120 – 129	37	54
130 – 139	31	85
140 – 149	11	96
150 – 159	4	100
	100	

Half the total frequency is 50 so the median lies in the "120 – 129 lbs" group, and in the (50 – 17) = 33rd item within it.

$$\begin{aligned}\text{Median weight} &= 120 + \frac{33}{37} \times 10 \\ &= 120 + 8.9 \\ &= 128.9 \\ &= 129 \text{ lbs to the nearest lb}\end{aligned}$$

7. Take, $c = 50$, $x_o = 224.5$

Distribution of the Sizes of Shareholdings in DEF Plc

Number of shares held	Mid-value	No. of persons with these shares	$d = \frac{x - x_o}{c}$	fd	Cumulative frequency
Under 50	24.5	80	–4	–320	80
50 – 99	74.5	142	–3	–426	222
100 – 149	124.5	110	–2	–220	332
150 – 199	174.5	75	–1	–75	407
200 – 249	224.5	52	0	0	459
250 – 299	274.5	38	+1	+38	497
300 – 349	324.5	28	+2	+56	525
350 – 399	374.5	15	+3	+45	540
400 –449	424.5	9	+4	+36	549
450 – 499	474.5	7	+5	+35	556
500 – 549	524.5	3	+6	+18	559
550 – 599	574.5	2	+7	+14	561
Total		561		–1,041 +242 = –799	

(a) $\bar{x} = 224.5 + \frac{-799}{561} \times 50$

$= 224.5 - (1.424 \times 50)$

$= 224.5 - 71.2 = 153.3 =$ 153 shares to the nearest whole number.

(N.B. You may have chosen a different x_o, but the result will be the same.)

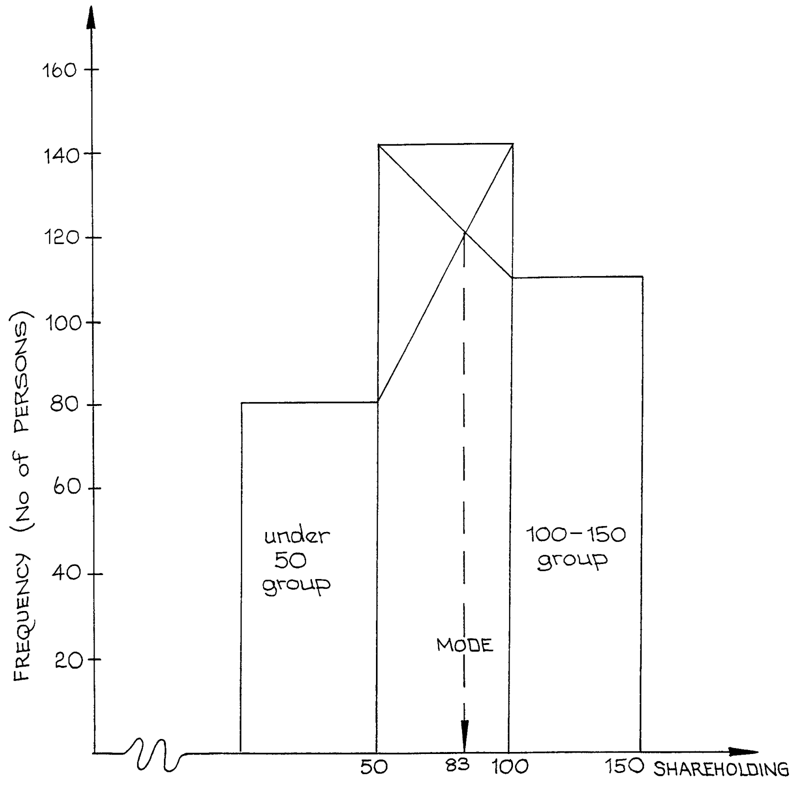
(b) Half of the total frequency is 280.5 and therefore the median lies in the "100 – 149" group and it is the (280.5 – 222 =) 58.5th item within it.

Median $= 99.5 + \frac{58.5 \times 50}{110} = 126.1$

$=$ 126 shares to the nearest whole number.

(c) The modal group is the "50 – 99" group and the estimated mode is given by drawing this group, together with the surrounding class groups, on a histogram, as shown on the next page.

Reading from this, the estimated mode = 83 shares.


FREQUENCY (No of PERSONS)
160
140
120
100
80
60
40
20
under 50 group
100–150 group
MODE
50
83
100
150
SHAREHOLDING

Study Unit 9

Measures of Dispersion

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INTRODUCTION

In order to get an idea of the general level of values in a frequency distribution, we have studied the various measures of location that are available. However, the figures which go to make up a distribution may all be very close to the central value, or they may be widely dispersed about it – for example, the mean of 49 and 51 is 50, but the mean of 0 and 100 is also 50. You can see, therefore, that two distributions may have the same mean, but the individual values may be spread about the mean in vastly different ways.

When applying statistical methods to practical problems, a knowledge of this spread (which we call "dispersion" or "variation") is of great importance. Examine the figures in the following table:

Table 9.1: Output from two factories over 10 weeks

Week	Weekly output	
	Factory A	Factory B
1	94	136
2	100	92
3	106	110
4	100	36
5	90	102
6	101	57
7	107	108
8	98	81
9	101	156
10	98	117
Total	995	995
Mean Output	99.5	99.5

Although the two factories have the same mean output, they are very different in their week-to-week consistency. Factory A achieves its mean production with only very little variation from week to week, whereas Factory B achieves the same mean by erratic ups-and-downs from week to week. This example shows that a mean (or other measure of location) does not, by itself, tell the whole story and we therefore need to supplement it with a "measure of dispersion".

As was the case with measures of location, there are several different measures of dispersion in use by statisticians. Each has its own particular merits and demerits, which will be discussed later. The measures in common use are:

- Range
- Quartile deviation
- Standard deviation

We will discuss each of these in this unit.

A. RANGE

This is the simplest measure of dispersion; it is simply the difference between the largest and the smallest. In the example just given, we can see that the lowest weekly output for Factory A was 90 and the highest was 107; the ***range*** is therefore 17. For Factory B the range is 156 – 36 = 120. The larger range for Factory B shows that it performs less consistently than Factory A.

The advantage of the range as a measure of the dispersion of a distribution is that it is very easy to calculate and its meaning is easy to understand. For these reasons it is used a great deal in industrial quality control work. Its disadvantage is that it is based on only two of the individual values and takes no account of all those in between. As a result, one or two extreme results can make it quite unrepresentative. Consequently, the range is not much used except in the case just mentioned.

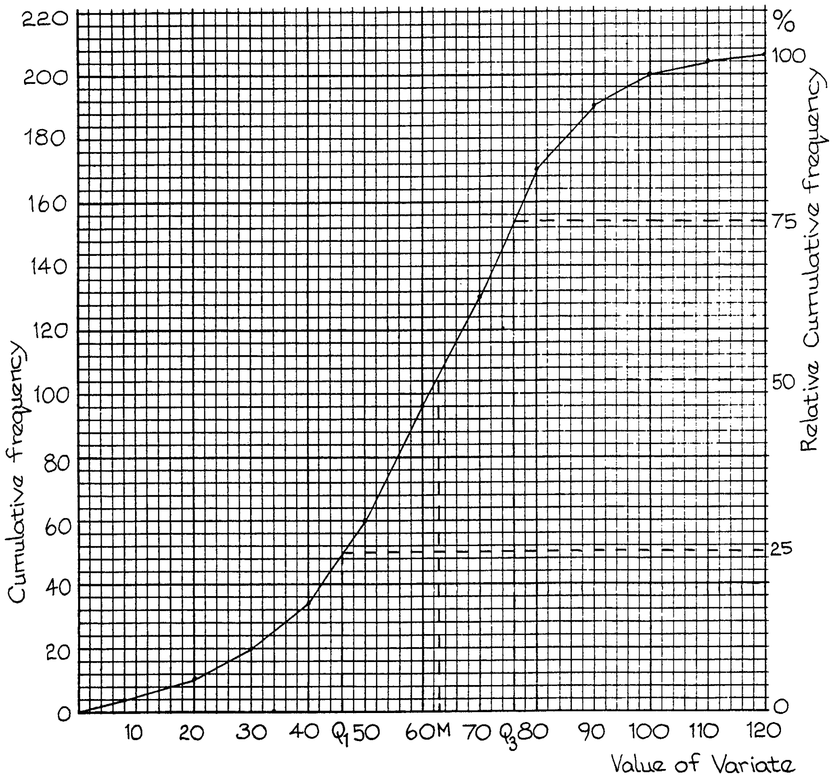
B. THE QUARTILE DEVIATION, DECILES AND PERCENTILES

The Quartile Deviation

This measure of dispersion is sometimes called the "***semi-interquartile range***".

To understand it, you must cast your mind back to the method of obtaining the median from the ogive. The median, you remember, is the value which divides the total frequency into two halves. The values which divide the total frequency into quarters are called ***quartiles*** and they can also be found from the ogive, as shown in Figure 9.1.

Figure 9.1: Identifying quartiles from an ogive



This is the same ogive that we drew the previous study unit when finding the median of a grouped frequency distribution.

You will notice that we have added the relative cumulative frequency scale to the right of the graph. 100% corresponds to 206, i.e. the total frequency. It is then easy to read off the values of the variate corresponding to 25%, 50% and 75% of the cumulative frequency, giving the lower quartile (Q_1), the median and the upper quartile (Q_3) respectively.

Q_1 = 46.5

Median = 63 (as found previously)

Q_3 = 76

The difference between the two quartiles is the ***interquartile range*** and half of the difference is the ***semi-interquartile range*** or ***quartile deviation***:

$$\text{Quartile deviation} = \frac{Q_3 - Q_1}{2}$$

$$= \frac{76 - 46.5}{2}$$

$$= \frac{29.5}{2}$$

$$= 14.75$$

Alternatively, you can work out:

25% of the total frequency, i.e. $\frac{206}{4} = 51.5$, and

75% of the total frequency, i.e. 154.5

and then read from the ogive the values of the variate corresponding to 51.5 and 154.5 on the cumulative frequency scale (i.e. the left-hand scale). The end result is the same.

Calculation of the Quartile Deviation

The quartile deviation is not difficult to calculate in situations where the graphical method is not acceptable. Remember that graphical methods are never quite as accurate as calculations.

We shall again use the same example.

The table of values is reproduced for convenience as Table 9.2:

Table 9.2: Data for determining the quartile deviation of a grouped frequency distribution

Group	Frequency (f)	Cumulative frequency (F)
0 – 10	4	4
10 – 20	6	10
20 – 30	10	20
30 – 40	16	36
40 – 50	24	60
50 – 60	32	92
60 – 70	38	130
70 – 80	40	170
80 – 90	20	190
90 – 100	10	200
100 – 110	5	205
110 – 120	1	206
Total	206	

The lower quartile is the 25% point in the total distribution and the upper quartile the 75% point.

As the total frequency is 206:

$$\text{the 25\% point} = \frac{206}{4} = 51\tfrac{1}{2} \text{ and}$$

$$\text{the 75\% point} = \frac{3}{4} \times 206 = 154\tfrac{1}{2}.$$

We can make the calculations in exactly the same manner as we used for calculating the median.

Looking at Table 9.2, the 51½th item comes in the 40 – 50 group and will be the (51½ – 36) = 15½th item within it.

$$\text{So, the lower quartile} = 40 + \frac{15\tfrac{1}{2}}{24} \times 10$$

$$= 40 + 6.458$$

$$= 46.458$$

Similarly, the upper quartile will be the 154½th item which is in the 70 – 80 group and is the (154½ – 130) = 24½th item within it.

$$\text{So, the upper quartile} = 70 + \frac{24\tfrac{1}{2}}{40} \times 10$$

$$= 70 + 6.125$$

$$= 76.125$$

Remember that the units of the quartiles and of the median are the same as those of the variate.

$$\text{The quartile deviation} = \frac{Q_3 - Q_1}{2}$$

$$= \frac{76.125 - 46.458}{2}$$

$$= \frac{29.667}{2}$$

$$= 14.8335$$

$$= 14.8 \text{ to 1 decimal place}$$

The quartile deviation is unaffected by an occasional extreme value. It is not based, however, on the actual value of all the items in the distribution and, to this extent, it is less representative than the standard deviation (which we discuss later). In general, when a median is the appropriate measure of location then the quartile deviation should be used as the measure of dispersion.

Deciles and Percentiles

It is sometimes convenient, particularly when dealing with wages and employment statistics, to consider values similar to the quartiles, but which divide the distribution more finely. Such partition values are deciles and percentiles. From their names you will probably have guessed that the deciles are the values which divide the total frequency into tenths and the percentiles are the values which divide the total frequency into hundredths. Obviously it is only meaningful to consider such values when we have a large total frequency.

The deciles are labelled D_1, D_2 ... D_9. The second decile D_2, then, is the value below which 20% of the data lies and the sixth decile, D_6, is the value below which 60% of the data lies.

The percentiles are labelled P_1, P_2 ... P_{99}. Again we can say that, for example, P_5 is the value below which 5% of the data lies and P_{64} is the value below which 64% of the data lies.

Using the same example as above, let us calculate, as an illustration, the third decile D_3. The method follows exactly the same principles as the calculation of the median and quartiles.

The total frequency is 206 and D_3 is the value below which 30% of the data lies:

$$30\% \text{ of } 206 \text{ is } \frac{30}{100} \times 206 = 61.8$$

We are, therefore, looking for the value of the 61.8th item. A glance at the cumulative frequency column shows that the 61.8th item lies in the 50 – 60 group, and is the (61.8 – 60) = 1.8th item within it. So:

$$D_3 = 50 + \frac{1.8}{32} \times 10$$

$$= 50 + \frac{18}{32}$$

$$= 50.6 \text{ to 1 dec. place}$$

Therefore. 30% of our data lies below 50.6.

We could also have found this result graphically – check that you agree with the calculation by reading D_3 from the graph in Figure 9.1. You will see that the calculation method enables us to give a more precise answer than is obtainable graphically.

C. STANDARD DEVIATION

The most important of the measures of dispersion is the standard deviation. Except for the use of the range in statistical quality control and the use of the quartile deviation in wages statistics, the standard deviation is used almost exclusively in statistical practice.

The standard deviation is defined as the ***square root of the variance*** and so we need to know how to calculate the variance first.

The Variance and Standard Deviation

We start by finding the deviations from the mean, and then squaring them, which removes the negative signs in a mathematically acceptable fashion. The variance is then defined as the mean of the deviations from the mean, squared:

$$\text{Variance} = \frac{\Sigma(x - \bar{x})^2}{n}, \text{ where n is the total frequency.}$$

Consider the following example.

Table 9.3: Output from factory A over 10 weeks

Week	Weekly output (x)	Deviation $(x - \bar{x})$	Deviation squared $(x - \bar{x})^2$
1	94	–5.5	30.25
2	100	+0.5	0.25
3	106	+6.5	42.25
4	100	+0.5	0.25
5	90	–9.5	90.25
6	101	+1.5	2.25
7	107	+7.5	56.25
8	98	–1.5	2.25
9	101	+1.5	2.25
10	98	–1.5	2.25
Total	995	+18.0 –18.0	228.50
Mean Output	99.5	Variance =	22.85

The mean of the squared deviations is the variance:

$$\text{Variance} = \frac{\Sigma(x - \bar{x})^2}{n}$$

$$= \frac{228.50}{10}$$

$$= 22.85$$

The formula for the standard deviation, which is defined as the square root of the variance, is:

$$SD = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$, again where n is the total frequency.

In this example, then:

$$SD = \sqrt{22.85} = 4.78.$$

The symbol usually used to denote standard deviation is "σ" (theta), although you may sometimes find "s" or "sd" used.

Alternative Formulae for Standard Deviation and Variance

The formula we have used to calculate the standard deviation is:

$$\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} \qquad \textit{Formula (a)}$$

where: x = value of the observation

$\bar{x}$ = mean of the observations

n = number of observations

As the standard deviation is defined as the square root of the variance, therefore:

$$\text{Variance} = \sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n}$$

By expanding the expression $\sum(x-\bar{x})^2$ we can rewrite the formula for standard deviation in two alternative and sometimes more useful forms, as follows:

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \qquad \textit{Formula (b)}$$

or $$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} \qquad \textit{Formula (c)}$$

The choice of the formula to use depends on the information that you already have and the information that you are asked to give. In any calculation, you should aim to keep the arithmetic as simple as possible and the rounding errors as small as possible.

- If you already know $\bar{x}$ and it is a small integer, there is no reason why you should not use formula *(a)*.
- If you already know $\bar{x}$ but it is not an integer (as in the example above), then formula *(b)* is the best to use.
- If you do not know $\bar{x}$, then you should use formula *(c)* – particularly if you are not asked to calculate $\bar{x}$.

When you are dealing with data in the form of a simple or grouped frequency distribution, then the formula for calculating the standard deviation needs to be expressed slightly differently:

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\sum f}} \qquad \textit{Formula (d)}$$

or $\sigma = \sqrt{\frac{1}{n}\left(\sum fx^2 - \frac{(\sum fx)^2}{n}\right)}$, *where* $n = \Sigma f$ *Formula (e)*

Formula *(d)*, like formula *(a)*, is derived directly from the definition of the standard deviation. Formula *(e)* is obtained by using the sigma notation as in formula *(c)* and this is the one to use in calculations, as shown in the following sections.

Standard Deviation of a Simple Frequency Distribution

If the data in Table 9.3 had been given as a frequency distribution (as is often the case), then only the different values would appear in the "x" column and we would have to remember to multiply each result by its frequency. This is shown in Table 9.4 where we set out the information for calculating the standard deviation by applying formula *(d)*:

Table 9.4: Determining the standard deviation of a simple frequency distribution

Weekly output (x)	Frequency (f)	fx	Deviation $(x - \bar{x})$	Deviation squared $(x - \bar{x})^2$	$f(x - \bar{x})^2$
90	1	90	–9.5	90.25	90.25
94	1	94	–5.5	30.25	30.25
98	2	196	–1.5	2.25	4.50
100	2	200	+0.5	0.25	0.50
101	2	202	+1.5	2.25	4.50
106	1	106	+6.5	42.25	42.25
107	1	107	+7.5	56.25	56.25
Total	10	995			228.50
Mean		99.5		Variance =	22.85

The standard deviation is then worked out in the same way as previously, as follows:

$$\sigma = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{\sum f}}$$

$$= \sqrt{\frac{228.5}{10}}$$

$$= \sqrt{22.85} \text{ (the variance)}$$

$$= 4.78.$$

However, we could have applied formula *(e)* and reduced the calculation as follows:

Table 9.5: Determining the standard deviation of a simple frequency distribution

Weekly output (x)	Frequency (f)	fx	x^2	fx^2
90	1	90	8,100	8,100
94	1	94	8,836	8,836
98	2	196	9,604	19,208
100	2	200	10,000	20,000
101	2	202	10,201	20,402
106	1	106	11,236	11,236
107	1	107	11,449	11,449
	10	995		99,231

Therefore: $n = \Sigma f = 10$

$\Sigma fx = 995$

$\Sigma fx^2 = 99{,}231$

Calculating the standard deviation proceeds as follows:

$$\sigma = \sqrt{\frac{1}{n}\left(\sum fx^2 - \frac{(\sum fx)^2}{n}\right)}$$

$$= \sqrt{\frac{1}{10}\left(99{,}231 - \frac{995^2}{10}\right)}$$

$$= \sqrt{\frac{1}{10}\left(99{,}231 - \frac{990{,}025}{10}\right)}$$

$$= \sqrt{\frac{1}{10}(99{,}231 - 99{,}002.5)}$$

$$= \sqrt{\frac{1}{10} \times 228.5}$$

$$= \sqrt{22.85} \text{ (the variance)}$$

$$= 4.78.$$

Standard Deviation of a Grouped Frequency Distribution

When we come to the problem of finding the standard deviation of a grouped frequency distribution, we again assume that all the readings in a given group fall at the mid-point of the group. Using formula *(e)* and applying it to the distribution shown in Table 9.6, the standard deviation is calculated as follows.

Table 9.6: Determining the standard deviation of a grouped frequency distribution

Class	Mid-value (x)	Frequency (f)	fx	x^2	fx^2
10 – 20	15	2	30	225	450
20 – 30	25	5	125	625	3,125
30 – 40	35	8	280	1,225	9,800
40 – 50	45	6	270	2,025	12,150
50 – 60	55	5	275	3,025	15,125
60 – 70	65	3	195	4,225	12,675
70 – 80	75	1	75	5,625	5,625
Total		30	1,250		58,950

Therefore: $n = \Sigma f = 30$

$\Sigma fx = 1{,}250$

$\Sigma fx^2 = 58{,}950$

Calculating the standard deviation proceeds as follows:

$$\sigma = \sqrt{\frac{1}{n}\left(\sum fx^2 - \frac{(\sum fx)^2}{n}\right)}$$

$$= \sqrt{\frac{1}{30}\left(58{,}950 - \frac{1{,}250^2}{30}\right)}$$

$$= \sqrt{\frac{1}{30}\left(58{,}950 - \frac{1{,}562{,}500}{30}\right)}$$

$$= \sqrt{\frac{1}{30}(58{,}950 - 52{,}083.33)}$$

$$= \sqrt{\frac{1}{30} \times 6{,}866.67}$$

$$= \sqrt{228.89} \quad \text{(the variance)}$$

$$= 15.13.$$

The arithmetic for this table is rather tedious even with an calculator, but we can extend the "short-cut" method which we used for finding the arithmetic mean of a distribution, to find the standard deviation as well. In that method we:

- Worked from an assumed mean.
- Worked in class intervals.
- Applied a correction to the assumed mean.

Table 9.7 applies this approach to working out the standard deviation.

Table 9.7: Determining the standard deviation of a grouped frequency distribution (short cut)

$x_o = 35, \; c = 10$

Class	Mid-value (x)	Frequency (f)	$d = \frac{x - x_o}{c}$	fd	d^2	fd^2
10 – 20	15	2	–2	–4	4	8
20 – 30	25	5	–1	–5	1	5
30 – 40	35	8	0	0	0	0
40 – 50	45	6	+1	+6	1	6
50 – 60	55	5	+2	+10	4	20
60 – 70	65	3	+3	+9	9	27
70 – 80	75	1	+4	+4	16	16
Total		30		+29 –9		82
				+20		

The standard deviation is calculated in four steps from this table, as follows:

(a) The approximate variance is obtained from $\frac{1}{n} \times \Sigma fd^2$ which in our case is equal to

$$\frac{1}{30} \times 82 = \frac{82}{30} = 2.7333$$

(b) The correction is $-\left(\frac{1}{n} \times \Sigma fd\right)^2$

It is always ***subtracted*** from the approximate variance to get the corrected variance. In our case the correction is:

$$-(20/30)^2 = -0.4444.$$

(c) The ***corrected variance*** is thus:

$$2.7333 - 0.4444 = 2.2889$$

(d) The ***standard deviation*** is then the class interval times the square root of the corrected variance:

$$SD = \sqrt{2.2889} = 1.513 \text{ class intervals} \times 10$$
$$= 15.13$$

This may seem a little complicated, but if you work through the example a few times, it will all fall into place. Remember the following points:

- Work from an assumed mean at the mid-point of any convenient class.
- The correction is always subtracted from the approximate variance.
- As you are working in class intervals, it is necessary to multiply by the class interval as the last step.
- The correction factor is the same as that used for the "short-cut" calculation of the mean, but for the SD it has to be squared.

- The column for d^2 may be omitted since fd^2 = fd multiplied by d. But do not omit it until you have really grasped the principles involved.
- The formula for calculating the standard deviation in this way can be written as:

$$SD = c\sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2}$$

- The assumed mean should be chosen from a group with the most common interval and c will be that interval. If the intervals vary too much, we revert to the basic formula.

Characteristics of the Standard Deviation

In spite of the apparently complicated method of calculation, the standard deviation is the measure of dispersion used in all but the very simplest of statistical studies. It is based on all of the individual items, it gives slightly more emphasis to the larger deviations but does not ignore the smaller ones and, most important, it can be treated mathematically in more advanced statistics.

D. THE COEFFICIENT OF VARIATION

Suppose that we are comparing the profits earned by two businesses. One of them may be a fairly large business with average monthly profits of £50,000, while the other may be a small firm with average monthly profits of only £2,000. Clearly, the general level of profits is very different in the two cases, but what about the month-by-month variability? We can compare the two firms as to their variability by calculating the two standard deviations.

Let us suppose that both standard deviations come to £500. Now, £500 is a much more significant amount in relation to the small firm than it is in relation to the large firm so that, although they have the same standard deviations, it would be unrealistic to say that the two businesses are equally consistent in their month-to-month earnings of profits.

To overcome this difficulty, we can express the SD as a percentage of the mean in each case. We call the result the "***coefficient of variation***".

Applying the idea to the figures which we have just quoted, we get coefficients of variation (usually indicated in formulae by V or CV) as follows:

For the large firm: $V = \frac{500}{50,000} \times 100 = 1.0\%$

For the small firm: $V = \frac{500}{2,000} \times 100 = 25.0\%$

This shows that, relatively speaking, the small firm is more erratic in its earnings than the large firm.

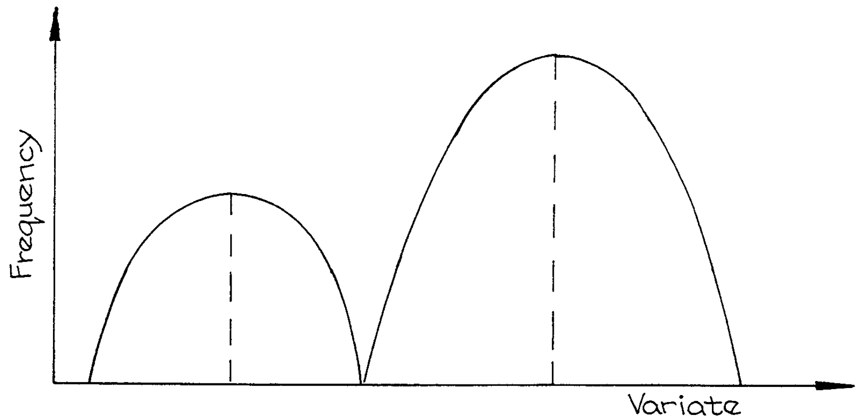
Note that although a standard deviation has the same units as the variate, the coefficient of variation is a ratio and thus has no units.

Another application of the coefficient of variation comes when we try to compare distributions the data of which are in different units – for example, when we try to compare a French business with an American business. To avoid the trouble of converting the dollars to francs (or vice versa) we can calculate the coefficients of variation in each case and thus obtain comparable measures of dispersion.

E. SKEWNESS

When the items in a distribution are dispersed equally on each side of the mean, we say that the distribution is symmetrical. Figure 9.2. shows two symmetrical distributions.

Figure 9.2: Symmetrical distributions

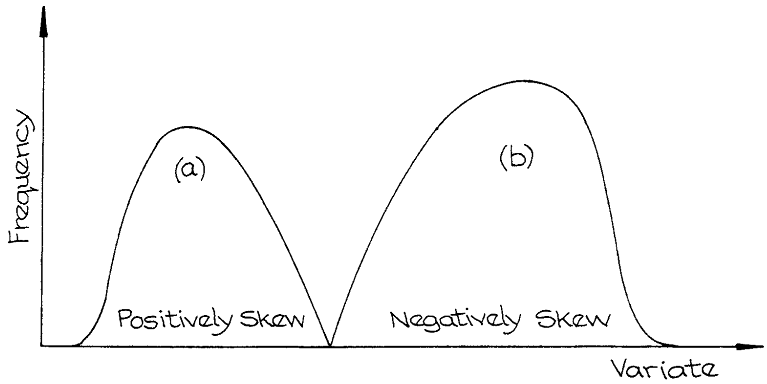


When the items are not symmetrically dispersed on each side of the mean, we say that the distribution is ***skewed*** or asymmetric.

A distribution which has a tail drawn out to the right is said to be ***positively skewed***, while one with a tail to the left, is ***negatively skewed***.

Two distributions may have the same mean and the same standard deviation but they may be differently skewed. This will be obvious if you look at one of the skew distributions in Figure 9.3 and then look at the **same one** through from the other side of the paper!

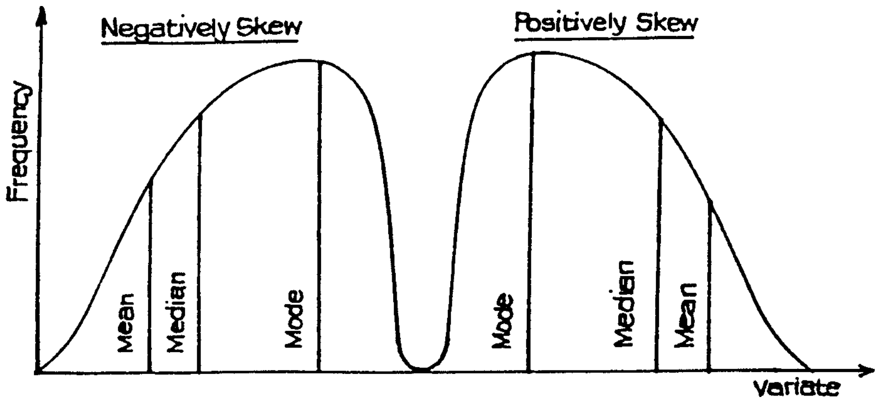
Figure 9.3: Skewed distributions



What, then, does skewness tell us? It tells us that we are to expect a few unusually high values in a positively skew distribution or a few unusually low values in a negatively skew distribution.

If a distribution is symmetrical, the mean, mode and median all occur at the same point, i.e. right in the middle. But in a skew distribution, the mean and the median lie somewhere along the side of the "tail", although the mode is still at the point where the curve is highest. The more skew the distribution, the greater the distance from the mode to the mean and the median, but these two are always in the same order; working outwards from the mode, the median comes first and then the mean – see Figure 9.4.

Figure 9.4: Positions of the mean, mode and median in skew distributions



For most distributions, except for those with very long tails, the following approximate relationship holds:

$$\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$$

The more skew the distribution, the more spread out are these three measures of location, and so we can use the amount of this spread to measure the amount of skewness. The most usual way of doing this is to calculate ***Pearson's First Coefficient of Skewness***:

$$\text{Pearson's First Coefficient of Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{SD}}$$

As we have seen, however, the mode is not always easy to find and so we can use an equivalent formula for ***Pearson's Second Coefficient of Skewness***:

$$\text{Pearson's Second Coefficient of Skewness} = \frac{3(\text{Mean} - \text{Median})}{\text{SD}}$$

You need to be familiar with these formulae as the basis of calculating the skewness of a distribution (or its "***coefficient of skewness***", as it is also called). When you do the calculation, remember to get the correct sign (+ or –) when subtracting the mode or median from the mean and then you will get negative answers from negatively skew distributions, and positive answers for positively skew distributions.

The value of the coefficient of skewness is between –3 and +3, although values below –1 and above +1 are rare and indicate very skew distributions.

Examples of variates with positive skew distributions include size of incomes of a large group of workers, size of households, length of service in an organisation and age of a workforce. Negative skew distributions occur less frequently. One such example is the age at death for the adult population in the UK.

Practice Questions

1. (a) Explain the advantages of using the coefficient of variation as a measure of dispersion.

   (b) A small finance company has loaned money to 10,000 customers. Each loan is a multiple of £1,000 and the details are given below:

Loan Amount *(£000)*	Number of Customers
1	535
2	211
3	427
4	1,360
5	1,650
6	1,504
7	1,280
8	1,615
9	1,012
10	406
11 or more	0
	10,000

Calculate:

(i) The arithmetic mean of the loan amounts (£).

(ii) The standard deviation of the loan amounts (£).

(iii) The coefficient of variation of the loan amounts.

2. Calculate the arithmetic mean and standard deviation for the following data using the "short-cut" method. Explain carefully what is measured by the standard deviation.

Income of borrower £	No. of borrowers
Under 1,800	1
1,800 – 2,400	6
2,400 – 3,000	9
3,000 – 3,600	14
3,600 – 4,200	17
4,200 – 4,800	23
4,800 – 5,400	28
5,400 – 6,000	35
6,000 – 6,600	39
6,600 – 7,200	48

3. Consider the following table:

Percentages of Owners with Assets Covered by Estate Duty Statistics

Assets over *(£)*	Assets not over *(£)*	Percentages	
		Year 1	Year 11
–	1,000	48.3	18.8
1,000	3,000	29.5	25.1
3,000	5,000	10.9	11.9
5,000	10,000	6.2	21.3
10,000	25,000	3.4	17.9
25,000	100,000	1.5	4.4
100,000	–	0.2	0.6
Total number of owners (000s)		18,448	19,140

You are required to:

(a) Derive by graphical methods the median, lower and upper quartile values for Year 1 and Year 11.

(b) Calculate for the two distributions the median values and explain any difference between these results and those obtained in (a).

(c) Estimate for each year the number of wealth owners with less than (i) £2,500, and (ii) £25,000.

(d) Write a brief report on the data, using your results where relevant.

4. Items are produced to a target dimension of 3.25 cm on a single machine. Production is carried out on each of three shifts, each shift having a different operator – A, B and C. It is decided to investigate the accuracy of the operators. The results of a sample of 100 items produced by each operator are as follows:

Accuracy of operators

Operator	Dimension of item *(cm)*								
	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29
A	1	6	10	21	36	17	6	2	1
B	0	0	7	25	34	27	6	1	0
C	3	22	49	22	4	0	0	0	0

Required:

(a) State, with brief explanation, which operator is:

(i) The most accurate in keeping to the target.

(i) The most consistent in his results.

(b) Illustrate your answers to part (a) by comparing the results obtained by each operator as frequency polygons on the same graph.

(c) Why is it that a measure of the mean level of the dimension obtained by each operator is an inadequate guide to his performance?

(d) Calculate the semi-interquartile range of the results obtained by operator A, to 3 decimal places.

5. Calculate the coefficient of skewness for a distribution with a mean of 10, median of 11, and standard deviation of 5. What does your answer tell you about this frequency distribution?

Now check your answers with the ones given at the end of the unit.

ANSWERS TO QUESTIONS FOR PRACTICE

1. (a) The coefficient of variation is the standard deviation expressed as a percentage of the arithmetic mean. It enables the ***relative*** variation of two sets of data to be compared. This is of use when comparing sets of data which have widely differing means, a situation which makes absolute comparisons difficult. The coefficient of variation has no units and thus the relative variation of sets of data expressed in different units can be compared. This is particularly useful when dealing, say, with profits expressed in different currencies.

(b) We need to draw up the following table:

Loan amount *(£000)* (x)	Number of customers (f)	fx	$(x - \bar{x})$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
1	535	535	–5	25	13,375
2	211	422	–4	16	3,376
3	427	1,281	–3	9	3,843
4	1,360	5,440	–2	4	5,440
5	1,650	8,250	–1	1	1,650
6	1,504	9,024	0	0	0
7	1,280	8,960	1	1	1,280
8	1,615	12,920	2	4	6,460
9	1,012	9,108	3	9	9,108
10	406	4,060	4	16	6,496
	10,000	60,000			51,028

(i) $\bar{x} = \dfrac{60{,}000}{10{,}000} = £6{,}000$

(ii) $\text{SD} = \sqrt{\dfrac{\Sigma f(x - \bar{x})^2}{\Sigma f}} = \sqrt{5.1028} = 2.259$ thousand pounds

$= £2{,}259$ to nearest £

(iii) Coefficient of variation $= \dfrac{2{,}259}{6{,}000} \times 100\% = 37.65\%$

2. We need to draw up the following table, with x_o = £5,100 and c = £600

Income of borrower £	Mid-value (x)	No. of borrowers (f)	$d = \frac{x - x_o}{c}$	fd	fd^2
Under 1,800	1,500 *	1	–6	–6	36
1,800 – 2,400	2,100	6	–5	–30	150
2,400 – 3,000	2,700	9	–4	–36	144
3,000 – 3,600	3,300	14	–3	–42	126
3,600 – 4,200	3,900	17	–2	–34	68
4,200 – 4,800	4,500	23	–1	–23	23
4,800 – 5,400	5,100	28	0	0	0
5,400 – 6,000	5,700	35	1	35	35
6,000 – 6,600	6,300	39	2	78	156
6,600 – 7,200	6,900	48	3	144	432
		220		86	1,170

* Assume the first class covers the range £1,200 – £1,800 so that its mid-value is £1,500.

$$\bar{x} = £5,100 + \frac{86}{220} \times 600$$

$$= £5,100 + 234.55$$

$$= £5,334.55$$

$= £5,335$ to the nearest £

$$\text{Variance} = \frac{1,170}{220} - \left(\frac{86}{220}\right)^2 \text{ in class interval units squared}$$

$$= 5.3182 - 0.1528$$

$$= 5.1654$$

$$\text{SD} = \sqrt{5.1654} = 2.2727 \text{ class interval units}$$

$$= £1,363.6$$

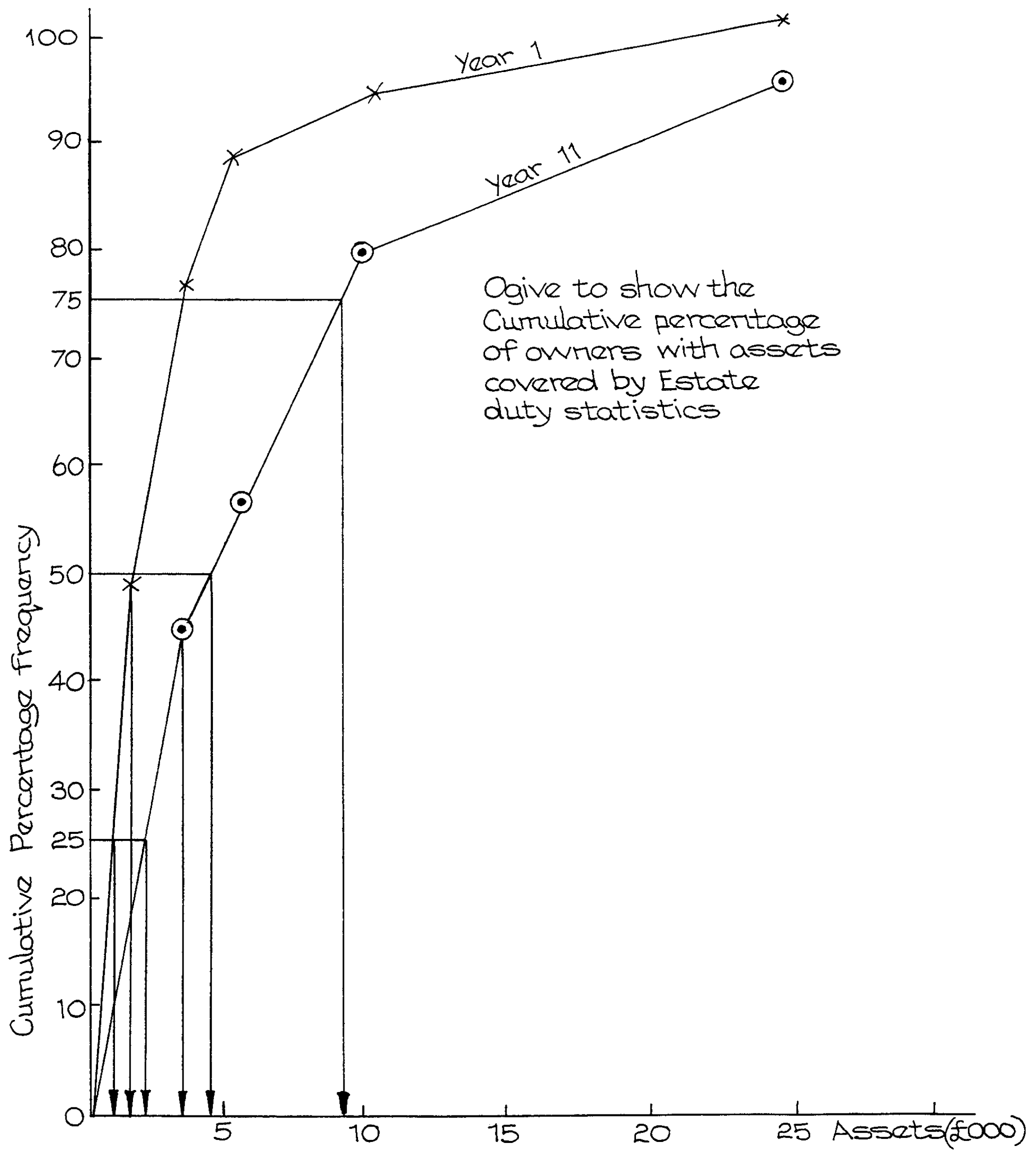
$= £1,364$ to the nearest £

The standard deviation measures the dispersion or spread of a set of data. It does this by taking every reading into account. We take the deviation of each reading from the mean and square it to remove any negative signs in a mathematically acceptable fashion. We then add all the squared deviations, find their mean and finally take the square root.

3 (a) We need firstly to draw up the following table

Assets over (£)	Cumulative % frequency (less than)	
	Year 1	Year 11
1,000	48.3	18.8
3,000	77.8	43.9
5,000	88.7	55.8
10,000	94.9	77.1
25,000	98.3	95.0
100,000	99.8	99.4

From this, we can draw ogives for both years to show the cumulative percentage of owners with assets of particular sizes:



Reading off the graph at a cumulative percentage frequency of 50 we obtain the median:

Year 1: Median = £1,100

Year 11: Median = £4,000

Reading off the graph at cumulative percentage frequencies of 25 and 75 we obtain the lower and upper quartile figures:

Year 1: Lower quartile = £500

Upper quartile = £2,800

Year 11: Lower quartile = £1,500

Upper quartile = £9,500

(b) *Year 1:*

The 50% mark is in the £1,000 – £3,000 group

$$\text{Median} = £1{,}000 + \frac{50 - 48.3}{29.5} \times £2{,}000$$

$$= £1{,}000 + \frac{1.7}{29.5} \times £2{,}000$$

$$= £1{,}000 + 115.25$$

$$= £1{,}115.25$$

Year 11:

The 50% mark is in the £3,000 – £5,000 group

$$\text{Median} = £3{,}000 + \frac{50 - 43.9}{11.9} \times £2{,}000$$

$$= £3{,}000 + \frac{6.1}{11.9} \times £2{,}000$$

$$= £3{,}000 + 1{,}025.21$$

$$= £4{,}025.21$$

These figures are more precise than the results in (a). This is to be expected as graphical methods are approximate and are limited by the size of the graph paper.

(c) Reading from the ogive at a value on the horizontal scale corresponding to £2,500, we find that in Year 1, 70% of owners had assets less than £2,500. In Year 11 the corresponding percentage was 37½.

Thus, the number of wealth owners with assets less than £2,500 was:

in Year 1: $\frac{70}{100} \times 18{,}448{,}000 = 12{,}913{,}600$

in Year 11: $\frac{37.5}{100} \times 19{,}140{,}000 = 7{,}177{,}500.$

From the table of values in (a), we note that in Year 1, 98.3% of owners had assets less than £25,000. In Year 11 the corresponding percentage was 95%.

Therefore the number of wealth owners with assets less than £25,000 was 18,134,384 in Year 1 and 18,183,000 in Year 11.

(d) The total number of owners with assets covered by estate duty statistics rose by 692,000 from Year 1 to Year 11.

As well as an increase in total wealth, there appears to have been some redistribution of wealth in this period. Whereas the modal group in Year 1 was under £1,000, in Year 11 it was £1,000 – £3,000 closely followed by the £5,000 – £10,000 group.

In Year 1, 1.7% had assets over £25,000 whereas 5% had assets over that amount in Year 11. Although inflation might be responsible for lifting some owners to higher groups, it is unlikely to account for a change in percentage as large as this.

The actual number of people with assets less than £25,000 was greater in Year 11 than in Year 1, presumably because more owners are covered by the statistics in Year 11. As a typical value for assets, we note that in Year 1, 50% of the owners had assets less than £1,115 but in Year 11, 50% had assets less than £4,025. The interquartile range for Year 1 was £500 – £2,800 but in Year 11 it was much larger, £1,500 – £9,500, indicating a greater spread of assets in the middle range in the later year.

4. (a) As this question says "with brief explanation", it implies that no calculations are to be made. We could, then, make the following observations:

- B is most accurate in keeping to the target of 3.25 cm as his items are spread about 3.25 cm, but with a narrower spread than A.
- C is the most consistent in his results as almost 50% of his items are 3.23 cm and his maximum spread either way is 0.02 cm.

You could back up your assertions with calculations of the mean and standard deviation in each case if time permits. It is good practice to work them out, so details of the calculations are now given.

You need to draw up the following table. Note that f_A and d_A refer to operator A. Similarly for operators B and C.

Dimension *(cm)* (x)	f_A	f_B	f_C	$d_A = d_B$	d_C	$f_A d_A$	$f_B d_B$	$f_C d_C$	$f_A d_A^2$	$f_B d_B^2$	$f_C d_C^2$
3.21	1	-	3	–4	–2	–4	-	–6	16	-	12
3.22	6	-	22	–3	–1	–18	-	–22	54	-	22
3.23	10	7	49	–2	0	–20	–14	0	40	28	0
3.24	21	25	22	–1	1	–21	–25	22	21	25	22
3.25	36	34	4	0	2	0	0	8	0	0	16
3.26	17	27	-	1	-	17	27	-	17	27	-
3.27	6	6	-	2	-	12	12	-	24	24	-
3.28	2	1	-	3	-	6	3	-	18	9	-
3.29	1	-	-	4	-	4	-	-	16	-	-
Total	100	100	100			–24	3	2	206	113	72

$$d_A = d_B = \frac{x - 3.25}{0.01} \qquad d_C = \frac{x - 3.23}{0.01}$$

$$\bar{x}_A = 3.25 + \frac{(-24)}{100} \times 0.01 = 3.25 - 0.0024 = 3.2476 \text{ cm}$$

$$\bar{x}_B = 3.25 + \frac{(+3)}{100} \times 0.01 = 3.25 + 0.0003 = 3.2503 \text{ cm}$$

$$\bar{x}_C = 3.23 + \frac{2}{100} \times 0.01 = 3.23 + 0.0002 = 3.2302 \text{ cm}$$

$$SD_A = 0.01\sqrt{\frac{206}{100} - \left(\frac{-24}{100}\right)^2} = 0.01\sqrt{2.06 - 0.0576}$$

$$= 0.01\sqrt{2.0024} = 0.0142 \text{ cm}$$

$$SD_B = 0.01\sqrt{\frac{113}{100} - \left(\frac{3}{100}\right)^2} = 0.01\sqrt{1.13 - 0.009}$$

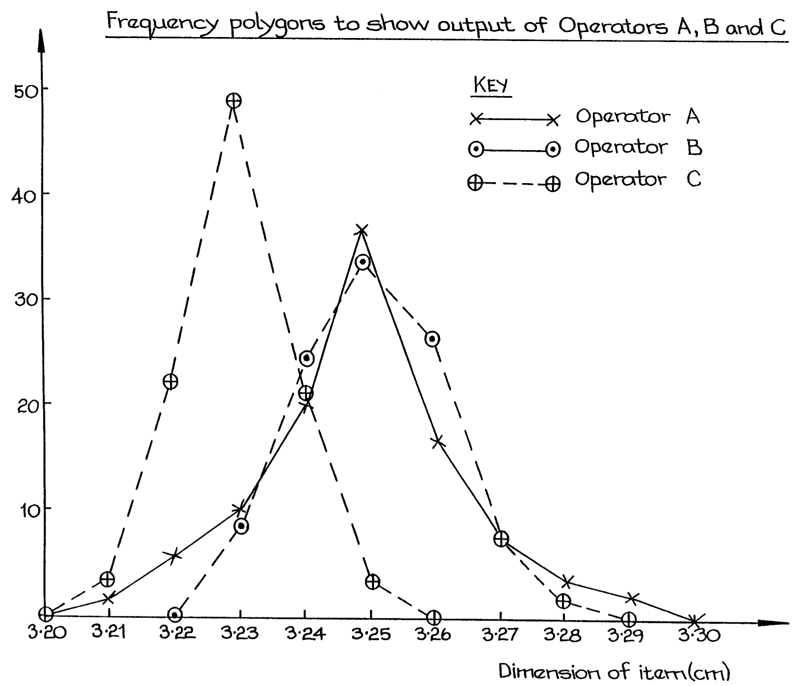
$$= 0.01\sqrt{1.1291} = 0.0106 \text{ cm}$$

$$SD_C = 0.01\sqrt{\frac{72}{100} - \left(\frac{2}{100}\right)^2} = 0.01\sqrt{0.72 - 0.0004}$$

$$= 0.01\sqrt{0.7196} = 0.00848 \text{ cm}$$

This confirms our earlier assertion as $\bar{x}_B$ is the mean closest to 3.25 cm and C has the smallest standard deviation.

(b) ***Frequency polygons***



(c) The mean on its own gives no indication of the spread of the readings about the mean. The mean dimension produced by A is close to 3.25 cm, but his output varies from 3.21 cm to 3.29 cm.

(d) Assume that each dimension has been measured to the nearest 0.01 cm – i.e. we assume that the one item produced by A in the 3.21 cm category is, in fact, less than 3.215 cm.

Cumulative frequency	Less than *(cm)*
1	3.215
7	3.225
17	3.235
38	3.245
74	3.255
91	3.265
97	3.275
99	3.285
100	3.295

The lower quartile Q_1 is the value below which 25% of the data lies, i.e. it is the value of the 25th item.

$$\therefore \quad Q_1 = 3.235 + \frac{(25-17)}{21} \times 0.01 \text{ cm}$$

$$= 3.235 + \frac{8 \times 0.01}{21} \text{ cm}$$

$$= 3.2388 \text{ cm}$$

Similarly

$$Q_3 = 3.255 + \frac{(75-74)}{17} \times 0.01 \text{ cm}$$

$$= 3.255 + \frac{1}{17} \times 0.01 \text{ cm}$$

$$= 3.2556 \text{ cm}$$

$$\text{The semi-interquartile range} = \frac{Q_3 - Q_1}{2}$$

$$= \frac{0.0168}{2} \text{ cm}$$

$$= 0.008 \text{ cm to 3 dec. pl.}$$

This tells us something about the spread of the middle 50% of the data. It could be of use to the manufacturer for predicting the variation in A's production so that he could estimate possible wastage when items are made too far from the target dimension.

5. Coefficient of skewness $= \dfrac{3(\text{Mean} - \text{Median})}{\text{Standard deviation}}$

$$= \frac{3(10 - 11)}{5}$$

$$= -0.6$$

Thus we can say that the distribution is slightly negatively skewed – i.e. asymmetric with the peak of the curve being to the right.

Study Unit 10

Time Series

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INTRODUCTION

Businesses and governments use statistical analysis of information collected at regular intervals over extensive periods of time to plan future policies. For example, sales values or unemployment levels recorded at yearly, quarterly or monthly intervals are examined in an attempt to predict their future behaviour. Such sets of values observed at regular intervals over a period of time are called ***time series***.

The analysis of this data is a complex problem as many variable factors may influence the changes which take place.

The first step in analysing the data is to plot the observations on a scattergram, with time evenly spaced across the x axis and measures of the dependent variable forming the y axis. This can give us a good visual guide to the actual changes, but is very little help in showing the component factors causing these changes or in predicting future movements of the dependent variable.

Statisticians have, therefore, constructed a number of models to describe the behaviour of time series. These assume that changes over time in the variable being studied are caused by the variation of four main factors. This gives rise to a number of approaches to the analysis of the inter-relationships between these factors. We shall, then, start the unit by working through an example of a simple time series and consider the general structure of time series in terms of the four factors, and then go on to examine the main approaches to their analysis. We shall also consider the use of time series analysis in forecasting.

A. STRUCTURE OF A TIME SERIES

Consider a factory employing a number of people in producing a particular commodity, say thermometers. Naturally, at such a factory during the course of a year some employees will be absent for various reasons. The following table shows the number of days lost through sickness over the last five years. Each year has been broken down into four quarters of three months. We have assumed that the number of employees at the factory remained constant over the five years.

Table 10.1: Days Lost Through Sickness at a Thermometer Factory

Year	Quarter	Days Lost
1995	1	30
	2	20
	3	15
	4	35
1996	1	40
	2	25
	3	18
	4	45
1997	1	45
	2	30
	3	22
	4	55
1998	1	50
	2	32
	3	28
	4	60
1999	1	60
	2	35
	3	30
	4	70

The Scattergram

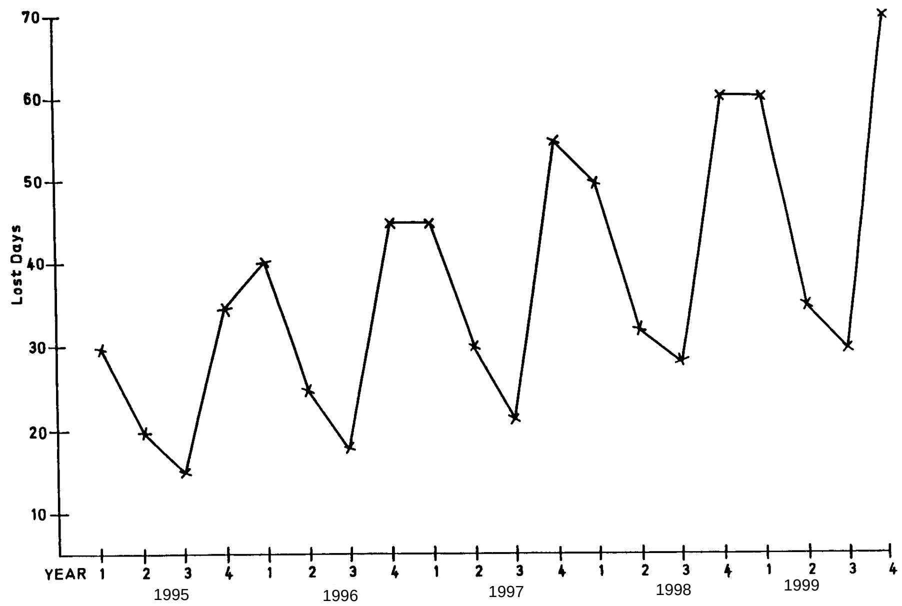
We will begin by plotting the scattergram for the data, as shown in Figure 10.1.

The scattergram of a time series is often called a **historigram**. (Do not confuse this with a histogram, which is a type of bar chart.) Note the following characteristics of a historigram:

- It is usual to join the points by ***straight*** lines. The only function of these lines is to help your eyes to see the pattern formed by the points.
- Intermediate values of the variables ***cannot*** be read from the historigram.
- A historigram is simpler than other scattergrams since no time value can have more than one corresponding value of the dependent variable.
- It is possible to change the scale of the y axis to highlight smaller changes.

Every historigram will look similar to this, depending on the scale of the y axis, but a careful study of the change of pattern over time will suggest which model should be used for analysis.

Figure 10.1: Scattergram of days lost through sickness (1996 – 1999)



There are four factors that influence the changes in a time series – trend, seasonal variations, cyclical fluctuations, and irregular or random fluctuations.

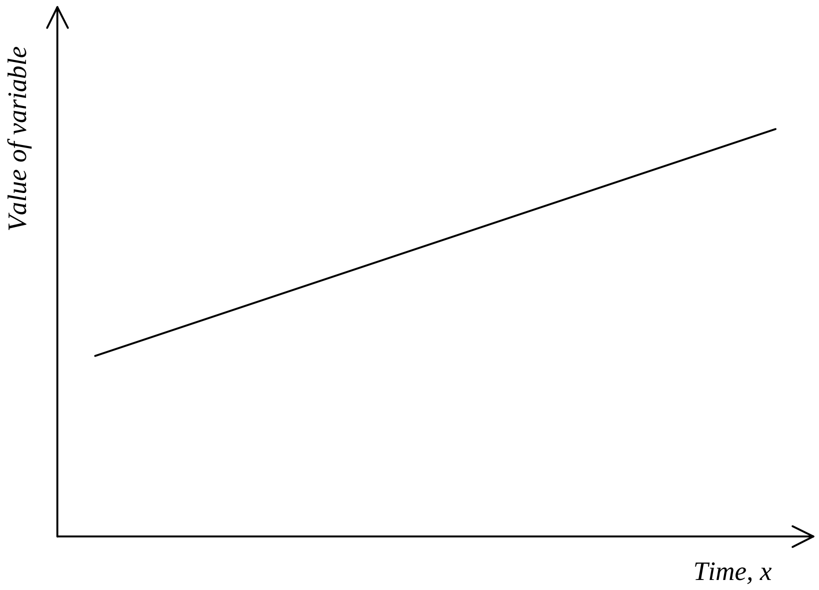
Trend

This is the change in general level over the whole time period and is often referred to as the ***secular trend***. You can see in Figure 10.1 that the trend is definitely upwards, in spite of the obvious fluctuations from one quarter to the next.

A trend can thus be defined as a ***clear tendency for the time series data to travel in a particular direction*** in spite of other large and small fluctuations.

An example of a linear trend is shown in Figure 9.2. There are numerous instances of a trend – for example, the amount of money collected from taxpayers is always increasing and, therefore, any time series describing government income from tax would show an upward trend.

Figure 10.2: Example of a linear trend line



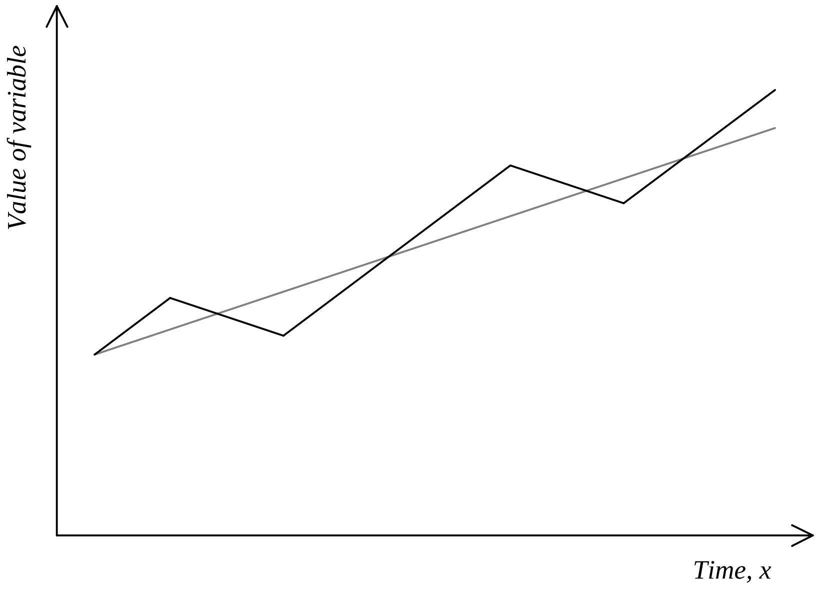
Seasonal Variations

These are variations which are ***repeated*** over relatively short periods of time. Those most frequently observed are associated with the seasons of the year – for example, ice-cream sales tend to rise during the summer months and fall during the winter months. You can see in our example of employees' sickness that more people are sick during the winter than in the summer.

If you can establish a pattern to the variation throughout the year, then this ***seasonal variation*** is likely to be similar from one year to the next. Thus, it would be possible to allow for it when estimating values of the variable in other parts of the time series. The usefulness of being able to calculate seasonal variation is obvious as, for example, it allows ice-cream manufacturers to alter their production schedules to meet these seasonal changes.

Figure 10.3 shows a typical seasonal variation that could apply to the examples above.

Figure 10.3: Example of seasonal variation

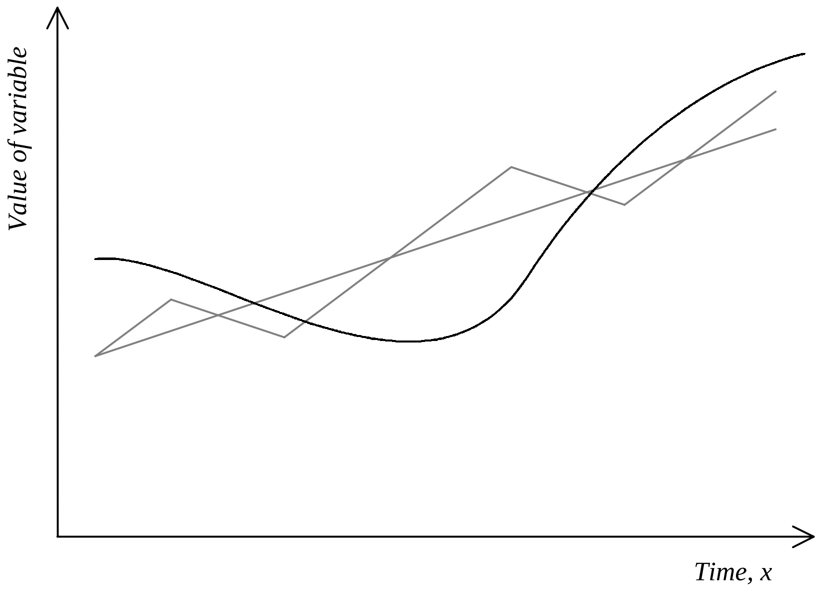


Cyclical Fluctuations

These are long-term, but fairly regular variations in the pattern of the variable. They are difficult to observe unless you have access to data over an extensive period of time during which external conditions have remained relatively constant. For example, it is well known in the textile trade that there is a cycle of about three years, during which time demand varies from high to low. This is similar to the phenomena known as the ***trade cycle*** which many economists say exists in the trading pattern of most countries, but for which there is no generally accepted explanation. It is also thought that weather patterns may be cyclical in nature over long periods.

Figure 10.4 shows how such a cyclical fluctuation would relate to an upward trend. In our example on sickness, a cyclical fluctuation could be caused by, say, a two-year cycle for people suffering from influenza.

Figure 10.4: Example of a cyclical fluctuation



As this type of fluctuation is difficult to determine, it is often considered with the final (fourth) element, and the two together are called the ***residual variation***.

Irregular or Random Fluctuations

Careful examination of Figure 10.1 shows that there are other relatively small irregularities which we have not accounted for and which do not seem to have any easily seen pattern. We call these irregular or random fluctuations and they may be due to errors of observation or to some one-off external influence which is difficult to isolate or predict. In our example there may have been a measles epidemic in 1998, but it would be extremely difficult to predict when and if such an epidemic would occur again.

B. ADDITIVE MODEL OF TIME SERIES ANALYSIS

To sum up, then, a time series (Y) can be considered as a combination of the following four factors:

- Trend (T)
- Seasonal variation (S)
- Cyclical fluctuation (C)
- Irregular fluctuations (I)

It is possible for the relationship between these factors and the time series to be expressed in a number of ways through the use of different mathematical models. Here, we shall consider in detail the ***additive model***.

The additive model, as its name suggests, can be expressed by the equation:

Time Series = Trend + Seasonal variation + Cyclical fluctuations + Random fluctuations

i.e. $Y = T + S + C + I$

As we noted, the cyclical and random fluctuations are often put together and called the "residual" (R), so we get:

$$Y = T + S + R$$

We shall only be concerned here with the trend, the other components being outside the scope of our studies at this time.

Trend

This is the ***most important factor*** of a time series.

Before deciding on the method to be used in finding it, we must decide whether the conditions that have influenced the series have remained stable over time. For example, if you have to consider the production of some commodity and want to establish the trend, you should first decide if there has been any significant change in conditions affecting the level of production, such as a sudden and considerable growth in the national economy. If there has, you must consider breaking the time series into sections over which the conditions have remained stable.

Having decided the time period you will analyse, you can use any one of the following methods to find the trend. The basic idea behind most of these methods is to average out the three other factors of variation so that you are left with the long-term trend.

(a) Graphical Method

Once you have plotted the historigram of the time series, it is possible to draw in by eye a line through the points to represent the trend. The result is likely to vary considerably from person to person, unless the plotted points lie very near to a straight line, so it is not a satisfactory method.

(b) Semi-Averages Method

This is a simple method which involves very little arithmetic. The time period is divided into equal parts, and the arithmetic means of the values of the dependent variable in each half are calculated. These means are then plotted at the quarter and three-quarters position of the time series. The line adjoining these two points represents the trend of the series. Note that this line will pass through the overall mean of the values of the dependent variable.

To calculate the mean of the first half of the time series, we need to divide the total of the values of the variable over that period by the number of individual time periods covered. In our example of days lost through sickness, the total time period covered is five years. The midpoint of the whole series is, then, mid-way between quarter 2 and quarter 3 of 1997.

Taking the data for this first half of the series, we get:

Year	Quarter	Days Lost
1995	1	30
	2	20
	3	15
	4	35
1996	1	40
	2	25
	3	18
	4	45
1997	1	45
	2	30
Totals	10 periods	303
	Mean	30.3

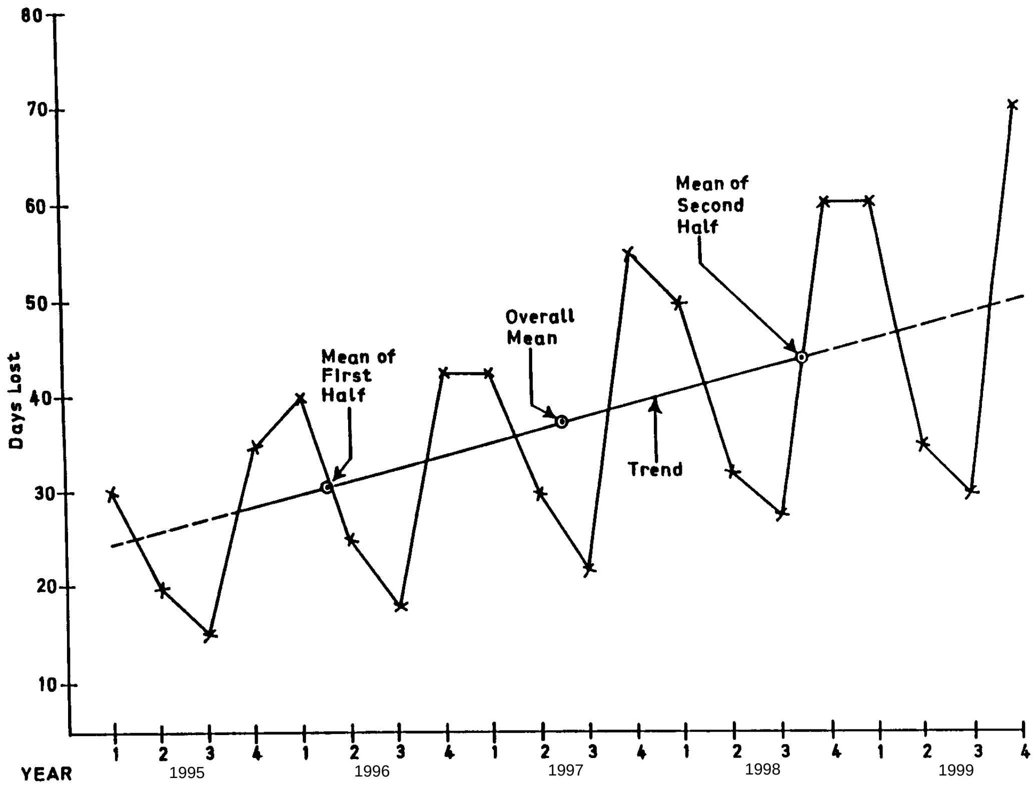
To calculate the mean of the second half of the period, we need to consider the data from the midpoint onwards:

Year	Quarter	Days Lost
1997	3	22
	4	55
1998	1	50
	2	32
	3	28
	4	60
1999	1	60
	2	35
	3	30
	4	70
Totals	10 periods	442
	Mean	44.2

These values are plotted on the historigram in Figure 10.5. You will notice that 30.3 days, as it is the mean for the first half, is plotted halfway between quarters 1 and 2 of 1996, and likewise 44.2 days is plotted halfway between quarters 3 and 4 of 1998. The trend line is then drawn between these two points and it can be extrapolated beyond these points as shown by the dotted line.

If there is an odd number of observations in the time series, the middle observation is ignored and the means of the observations on each side of it are calculated.

Figure 10.5: Trend line based on semi-averages method



(c) Moving Averages Method

So far, the methods we have discussed for finding trends have resulted in a straight line, but the actual trend may be a curve or a series of straight segments. The method of moving averages gives a way of calculating and plotting on the historigram a trend point corresponding to each observed point. These points are calculated by ***averaging a number of consecutive values*** of the dependent variable so that variations in individual observations are reduced. The number of consecutive values selected will depend on the length of the short-term or seasonal variation shown on the historigram.

The method of calculating a set of moving averages is illustrated by the following simple example.

Consider a time series comprising the seven numbers 6, 4, 5, 1, 9, 5, 6. If we take the number of time periods covered by the seasonal variations to be four (as in quarterly figures over each year), then a "moving average of order four" is needed. The calculation then proceeds as follows.

- Step 1: Find the average of the first to fourth numbers in the series:

$$\text{Average} = \frac{6+4+5+1}{4} = 4$$

- Step 2: Find the average of the second to fifth numbers in the series:

$$\text{Average} = \frac{4+5+1+9}{4} = 4.75$$

- Step 3: Find the average of the third to sixth numbers in the series:

$$\text{Average} = \frac{5+1+9+5}{4} = 5$$

- Step 4: Find the average of the fourth to seventh numbers in the series:

$$\text{Average} = \frac{1+9+5+6}{4} = 5.25$$

(If the series was longer, we could continue taking the average of the each successive set of four numbers in the series.)

Hence, in our simple example, the moving averages of order 4 are 4, 4.75, 5, 5.25, and these could be plotted on the historigram.

We selected a moving average of order 4 here since the data was quarterly. If the data was monthly, a moving average of order 12 would be needed, and for daily data, the order would be 7, and so on.

Returning now to our earlier example of days lost through sickness, we can calculate the trend values and plot them on Figure 10.6. A moving average of order 4 will again be used since the data is quarterly. The table of calculations is as follows.

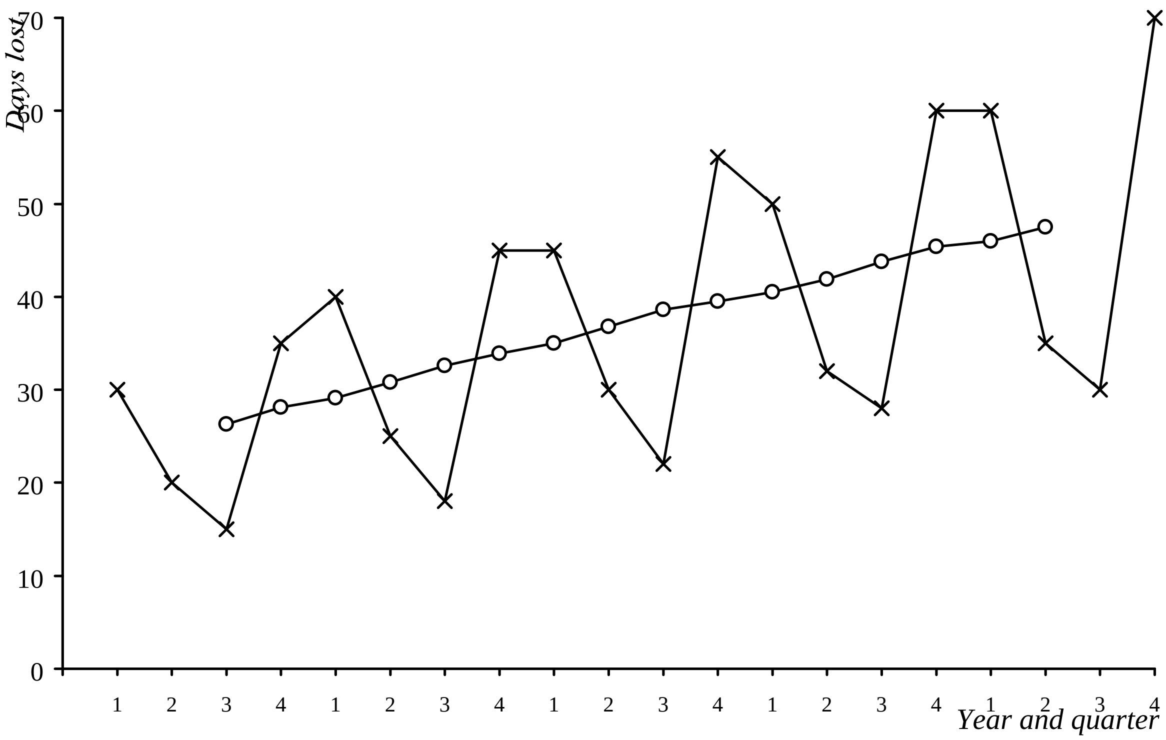
Table 10.2: Calculation table for moving averages method of determining trend

Year (1)	**Quarter** (2)	**Days Lost** (3)	**4-Quarter total** (4)	**Moving average** (5)	**Trend** (6)
1995	1	30			
	2	20			
			100	25.0	
	3	15			26.3
			110	27.5	
	4	35			28.1
			115	28.75	
1996	1	40			29.1
			118	29.5	
	2	25			30.8
			128	32.0	
	3	18			32.6
			133	33.25	
	4	45			33.9
			138	34.5	
1997	1	45			35.0
			142	35.5	
	2	30			36.8
			152	38.0	
	3	22			38.6
			157	39.25	
	4	55			39.5
			159	39.75	
1998	1	50			40.5
			165	41.25	
	2	32			41.9
			170	42.5	
	3	28			43.8
			180	45.0	
	4	60			45.4
			183	45.75	
1999	1	60			46.0
			185	46.25	
	2	35			47.5
			195	48.75	
	3	30			
	4	70			

Notice how the table of calculations is set out, with the numbers in columns (4) and (5) placed midway between the four quarterly readings to which they relate. This is because we are averaging over an even number of values, so the moving average would have to be plotted in this position on the historigram and would not correspond to any particular quarter. This makes it necessary to add column (6) which gives the mean of successive pairs of moving averages and these numbers are the trend values plotted in Figure 10.6 against particular quarters. (The values in column (6) are often called the centred moving averages.) The trend values are given correct to one decimal place as this is the greatest level of accuracy justified by the accuracy of the data.

If we were calculating a moving average with an odd number of values, it would not be necessary to carry out this final stage as the moving averages would be centred on an actual observation and so would be the trend values.

Figure 10.6: Moving average trend line



The main advantage of the moving averages method is that the trend values take into account the immediate changes in external factors which the trend line using the semi averages method is unable to do. However, this method has three disadvantages:

- The trend line cannot be found for the whole of the time series. As you can see from our example, there are no trend values for quarters at the beginning and end of the series.
- Problems can be encountered in deciding the order number, i.e. the period of fluctuation. Unless the seasonal or cyclical movement is definite and clear cut, the moving method of deriving the trend may yield a rather unsatisfactory line.
- Since the trend is calculated as a simple arithmetic mean it can be unduly influenced by a few extreme values.

C. FORECASTING

The reason for isolating the trend within a time series is to be able to make a prediction of its future values and thus estimate the movement of the time series.

In considering forecasting, there are two key assumptions which must be made at the outset.

(a) That conditions remain stable

Those conditions and factors which were apparent during the period over which the trend was calculated must be assumed to be unchanged over the period for which the forecast is made. If they do change, then the trend is likely to change with them, thus making any predictions inaccurate. For example, forecasts of savings trends based on given interest rates will not be correct if there is a sudden change either up or down in those rates.

(b) That extra factors will not arise

It is sometimes the case that, when trends are predicted beyond the limits of the data from which they are calculated, extra factors will arise which influence the trend. For example, there is a limit to the number of washing machines that can be sold within a country. This capacity is a factor that must be considered when making projections of the future sales of washing machines. Therefore, in forecasting from a time series, it must be assumed that such extra factors will not arise.

From the methods of defining trend considered in this unit, the moving averages method is the one used in forecasting .. Note that it is primarily concerned with short-term forecasts because the assumptions mentioned previously will break down gradually for periods of longer than about a year.

The method involves extending the moving average trend line drawn on the historigram of the time series. The trend line is extended by assuming that the gradient remains the same as that calculated from the data. The further forward you extend it, the more ***unreliable*** becomes the forecast.

Although fairly easy to calculate, this forecast must be treated with caution because it is based on the value of the trend calculated for the last period for which definitive data is available – if this happens to be an especially high or low value then it would influence the trend, and thus the forecast, considerably.

Practice Questions

1. A mortgage company had to write off the following bad debts for the years 1990 – 1996 inclusive:

Year	**Amount of bad debts** *(£000)*
1990	360
1991	390
1992	440
1993	500
1994	560
1995	600
1996	650

(a) Find the trend using the method of semi-averages.

(b) Plot the data on a historigram showing the trend line calculated in (a) above.

2. The following table shows sales of premium bonds at a large post office over five years. Calculate the trend using the moving averages method and show both the data and trendline on a historigram.

Year and Quarter		**Sales** *(£000)*
1995	1	320
	2	200
	3	150
	4	400
1996	1	400
	2	210
	3	180
	4	450
1997	1	800
	2	400
	3	340
	4	900
1998	1	950
	2	500
	3	400
	4	980

Now check your answers with the ones given at the end of the unit.

ANSWERS TO QUESTIONS FOR PRACTICE

1. (a) The mean of the first half $= \dfrac{(360 + 390 + 440)}{3}$

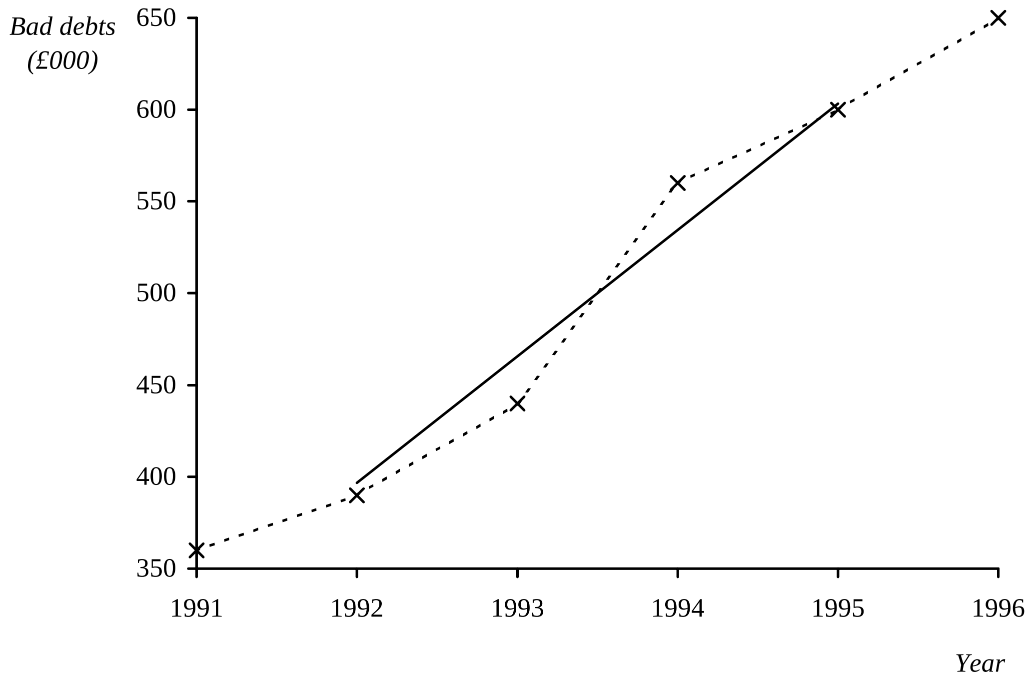
$= 396.7$

Mean of the second half $= \dfrac{560 + 600 + 650}{3}$

$= 603.3$

(The overall means are x = 4, y = 500)

(b) ***Historigram showing semi-averages trend line***

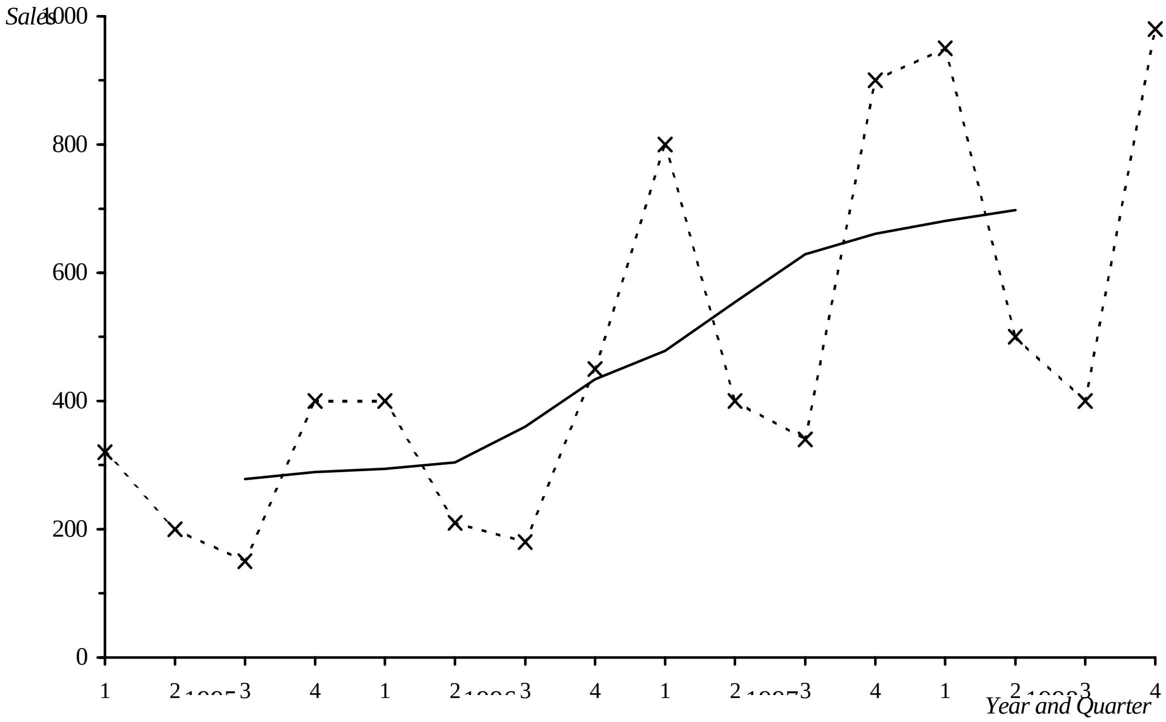


2. The calculation table for the trend is as follows.

Year and Quarter		Sales *(£000)*	4 Quarter Total	Moving Average	Trend
1995	1	320			
	2	200			
			1,070	268	
	3	150			278
			1,150	288	
	4	400			289
			1,160	290	
1996	1	400			294
			1,190	298	
	2	210			304
			1,240	310	
	3	180			360
			1,640	410	
	4	450			434
			1,830	458	
1997	1	800			478
			1,990	498	
	2	400			554
			2,440	610	
	3	340			629
			2,590	648	
	4	900			661
			2,690	673	
1998	1	950			681
			2,750	688	
	2	500			698
			2,830	708	
	3	400			
	4	980			

The historigram is shown on the next page.

Historigram showing moving averages trend line



Study Unit 11

Probability

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INTRODUCTION

"Probability" is one of those ideas about which we all have some notion, but which are not very definite. Initially, we will not spend our time trying to get an exact definition, but will confine ourselves to the task of grasping the idea generally and seeing how it can be used. Other words which convey much the same idea are "chance" and "likelihood".

Just as there is a scale of, say, temperature, because some things are hotter than others, so there is a scale of probability because some things are more probable than others – for example, snow is more likely to fall in winter than in summer, or a healthy person has more chance of surviving an attack of influenza than an unhealthy person. There is, you will note, some uncertainty in these matters. Most things in real life are uncertain to some degree or other, and it is for this reason that the theory of probability is of great ***practical*** value. It is the branch of mathematics which deals specifically with matters of uncertainty.

For the purpose of learning the theory, it is necessary to start with simple things like coin-tossing and dice-throwing, which may seem a bit remote from business and industrial life, but which will help you understand the more practical applications. We shall then move on to consider the application of probability to real situations.

The unit also introduces the concept of the normal distribution. This is a particular aspect of frequency distributions which we considered previously. The normal distribution has a number of properties which can be used to work out the probability of particular event happening.

A. ESTIMATING PROBABILITIES

Suppose someone tells you "there is a 50-50 chance that we will be able to deliver your order on Friday". This statement means something intuitively, even though when Friday arrives there are only two outcomes. Either the order will be delivered or it will not. Statements like this are trying to put probabilities or chances on uncertain events.

Probability is measured on a scale between 0 and 1. Any event which is impossible has a probability of 0, and any event which is certain to occur has a probability of 1. For example, the probability that the sun will not rise tomorrow is 0; the probability that a light bulb will fail sooner or later is 1. For uncertain events, the probability of occurrence is somewhere between 0 and 1. The 50-50 chance mentioned above is equivalent to a probability of 0.5.

Try to estimate probabilities for the following events. Remember that events which are more likely to occur than not have probabilities which are greater than 0.5, and the more certain they are the closer the probabilities are to 1. Similarly, events which are more likely **not** to occur have probabilities which are less than 0.5. The probabilities get closer to 0 as the events get more unlikely.

(a) The probability that a coin will fall heads when tossed.

(b) The probability that it will snow next Christmas.

(c) The probability that sales for your company will reach record levels next year.

(d) The probability that your car will not break down on your next journey.

(e) The probability that the throw of a die will show a six.

The probabilities are as follows:

(a) The probability of heads is one in two or 0.5.

(b) This probability is quite low. It is somewhere between 0 and 0.1.

(c) You can answer this one yourself.

(d) This depends on how frequently your car is serviced. For a reliable car it should be greater than 0.5.

(e) The probability of a six is one in six or $\frac{1}{6}$ or 0.167.

Theoretical Probabilities

Sometimes probabilities can be specified by considering the physical aspects of the situation.

For example, consider the tossing of a coin. What is the probability that it will fall heads? There are two sides to a coin. There is no reason to favour either side as a coin is symmetrical. Therefore, the probability of heads, which we call P(H) is:

$$P(H) = 0.5$$

Another example is throwing a die. A die has six sides. Again, assuming it is not weighted in favour of any of the sides, there is no reason to favour one side rather than another. Therefore, the probability of a six showing uppermost, P(6), is:

$$P(6) = \frac{1}{6} = 0.167$$

As a third and final example, imagine a box containing 100 beads of which 23 are black and 77 white. If we pick one bead out of the box at random (blindfold and with the box well shaken up), what is the probability that we will draw a black bead? We have 23 chances out of 100, so the probability is:

$$P(B) = \frac{23}{100} = 0.23$$

Probabilities of this kind, where we can assess them from our prior knowledge of the situation, are also called "a priori" probabilities.

In general terms, we can say that if an event E can happen in h ways out of a total of n possible equally likely ways, then the probability of that event occurring (called a success) is given by:

$$P(E) = \frac{\text{Number of possible ways of E occurring}}{\text{Total number of possible outcomes}}$$

$$= \frac{h}{n}$$

Empirical Probabilities

Often it is not possible to give a theoretical probability of an event.

For example, what is the probability that an item on a production line will fail a quality control test? This question can be answered either by measuring the probability in a test situation (i.e. empirically) or by relying on previous results. If 100 items are taken from the production line and tested, then:

$$\text{Probability of failure, P(F)} = \frac{\text{No. of items which fail}}{\text{Total no. of items tested}}$$

So, if 5 items actually fail the test

$$P(F) = \frac{5}{100} = 0.05$$

Sometimes it is not possible to set up an experiment to calculate an empirical probability. For example, what are your chances of passing a particular examination? You cannot sit a series of examinations to answer this. Previous results must be used. If you have taken 12 examinations in the past, and failed only one, you might estimate:

$$\text{Probability of passing, P(Pass)} = \frac{11}{12} = 0.92$$

B. TYPES OF EVENT

In probability, we can identify five types of event:

- Mutually exclusive
- Non-mutually-exclusive
- Independent
- Dependent or non-independent
- Complementary.

(a) Mutually exclusive events

If two events are mutually exclusive then the occurrence of one event precludes the possibility of the other occurring.

For example, the two sides of a coin are mutually exclusive since, on the throw of the coin, "heads" automatically rules out the possibility of "tails". On the throw of a die, a six excludes all other possibilities. In fact, all the sides of a die are mutually exclusive; the occurrence of any one of them as the top face automatically excludes any of the others.

(b) Non-mutually-exclusive events

These are events which can occur together. For example, in a pack of playing cards hearts and queens are non-mutually-exclusive since there is one card, the queen of hearts, which is both a heart and a queen and so satisfies both criteria for success.

(c) Independent events

These are events which are not mutually exclusive and where the occurrence of one event does not affect the occurrence of the other.

For example, the tossing of a coin in no way affects the result of the next toss of the coin; each toss has an independent outcome.

(d) Dependent or non-independent events

These are situations where the outcome of one event is dependent on another event.

For example, the probability of a car owner being able to drive to work in his car is dependent on him being able to start the car. The probability of him being able to drive to work given that the car starts is a ***conditional probability***. We can express this as:

P(Drive to work|Car starts)

where: the vertical line is a shorthand way of writing "given that".

(e) Complementary events

An event either occurs or it does not occur, i.e. we are certain that one or other of these situations holds.

For example, if we throw a die and denote the event where a six is uppermost by A, and the event where either a one, two, three, four or five is uppermost by $\overline{A}$ (or not A), then A and $\overline{A}$ are complementary, i.e. they are mutually exclusive with a total probability of 1. Thus:

$$P(A) + P(\overline{A}) = 1$$

This relationship between complementary events is useful as it is often easier to find the probability of an event not occurring than to find the probability that it does occur. Using the above formula, we can always find P(A) by subtracting $P(\overline{A})$ from 1.

C. THE TWO LAWS OF PROBABILITY

Addition Law for Mutually Exclusive Events

Consider again the example of throwing a die. You will remember that

$$P(6) = \tfrac{1}{6}$$

Similarly:

$$P(1) = \tfrac{1}{6}$$

$$P(2) = \tfrac{1}{6}$$

$$P(3) = \tfrac{1}{6}$$

$$P(4) = \tfrac{1}{6}$$

$$P(5) = \tfrac{1}{6}$$

What is the chance of getting 1, 2 or 3?

From the symmetry of the die you can see that P(1 or 2 or 3) = 0.5. But also, from the equations shown above you can see that:

$$P(1) + P(2) + P(3) = \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} = 0.5$$

This illustrates that

$$P(1 \text{ or } 2 \text{ or } 3) = P(1) + P(2) + P(3)$$

This result is a general one and it is called the ***addition law of probabilities for mutually exclusive events***. It is used to calculate the probability of one of any group of mutually exclusive events.

It is stated more generally as:

$$P(A \text{ or } B \text{ or } ... \text{ or } N) = P(A) + P(B) + ... + P(N)$$

where: A, B ... N are mutually exclusive events.

If ***all possible*** mutually exclusive events are listed, then it is certain that one of these outcomes will occur. For example, when the die is tossed there *must* be one number showing afterwards.

$$P(1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6) = 1$$

Using the addition law for mutually exclusive events, this can also be stated as

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

Again this is a general rule. The sum of the probabilities of a complete list of mutually exclusive events will always be 1.

Example

An urn contains 100 coloured balls. Five of these are red, seven are blue and the rest are white. One ball is to be drawn at random from the urn. What is the probability that it will be red?

$$\text{Probability that ball is red } P(R) = \tfrac{5}{100} = 0.05$$

What is the probability that it will be blue?

$$\text{Probability that ball is blue } P(B) = \tfrac{7}{100} = 0.07$$

What is the probability that it will be red or blue?

$$P(R \text{ or } B) = P(R) + P(B) = 0.05 + 0.07 = 0.12$$

This result uses the addition law for mutually exclusive events since a ball cannot be both blue and red.

What is the probability that it will be white?

The ball must be either red or blue or white. This is a complete list of mutually exclusive possibilities. Therefore:

$$P(R) + P(B) + P(W) = 1$$

Then the probability of one ball being white is given as:

$$\begin{aligned} P(W) &= 1 - P(R) - P(B) \\ &= 1 - 0.05 - 0.07 \\ &= 0.88. \end{aligned}$$

Addition Law for Non-Mutually-Exclusive Events

Events which are non-mutually-exclusive are, by definition, capable of occurring together.

To determine the probability of one of the events occurring, we can still use the addition law, but the probability of the events occurring together must be deducted:

$$P(A \text{ or } B \text{ or both}) = P(A) + P(B) - P(A \text{ and } B)$$

Example 1

If one card is drawn from a pack of 52 playing cards, what is the probability of (a) that it is either a spade or an ace, or (b) that it is either a spade or the ace of diamonds?

(a) Let event B be "the card is a spade" and event A be "the card is an ace".

We need to determine the probability of the event being either a spade or an ace. We also need to take account of the probability of the card being the ace of spades (i.e. the event being both A and B). However, this probability will be included in the probability of it being a spade and in the probability of it being an ace. Thus, we need to deduct that probability from one of the other probabilities. This is given by:

$$\text{P(spade or ace [or both])} = \text{P(A or B)}$$

$$= \text{P(A)} + \text{P(B)} - \text{P(A and B)}$$

$$\text{P(A)} = \frac{\text{No. of aces}}{\text{No. in pack}} = \frac{4}{52}$$

$$\text{P(B)} = \frac{\text{No. of spades}}{\text{No. in pack}} = \frac{13}{52}$$

$$\text{P(A and B)} = \frac{\text{No. of aces of spades}}{\text{No. in pack}} = \frac{1}{52}$$

$$\therefore \quad \text{P(spade or ace)} = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

(b) Let event B be "the card is a spade" and event A be "the card is the ace of diamonds".

Here, the conditions are not met by both events occurring, so the probability is given by:

$$\text{P(spade or ace of diamonds)} = \text{P(A)} + \text{P(B)}$$

$$\text{P(A)} = \frac{\text{No. of aces of diamonds}}{\text{No. in pack}} = \frac{1}{52}$$

$$\text{P(B)} = \frac{\text{No. of spades}}{\text{No. in pack}} = \frac{13}{52}$$

$$\therefore \quad \text{P(spade or ace of diamonds)} = \frac{1}{52} + \frac{13}{52} = \frac{14}{52} = \frac{7}{26}$$

Example 2

At a local shop 50% of customers buy unwrapped bread and 60% buy wrapped bread. What proportion of customers buy at least one kind of bread if 20% buy both wrapped and unwrapped.

Let: T represent those customers buying unwrapped bread

W represent those customers buying wrapped bread

Then:

$$\text{P(buy at least one kind of bread)} = \text{P(buy wrapped or unwrapped or both)}$$

$$= \text{P(T or W)}$$

$$= \text{P(T)} + \text{P(W)} - \text{P(T and W)}$$

$$= 0.5 + 0.6 - 0.2$$

$$= 0.9$$

So, $\frac{9}{10}$ of the customers buy at least one kind of bread.

Multiplication Law for Independent Events

Consider an item on a production line. This item could be defective or acceptable. These two possibilities are mutually exclusive and represent a complete list of alternatives.

Assume that:

Probability that it is defective, P(D) = 0.2

Probability that it is acceptable, P(A) = 0.8

Now consider another facet of these items. There is a system for checking them, but only every tenth item is checked. This is shown as:

Probability that it is checked, P(C) = 0.1

Probability that it is not checked, P(N) = 0.9

Again these two possibilities are mutually exclusive and they represent a complete list of alternatives. An item is either checked or it is not.

Consider the possibility that an individual item is both defective and not checked. These two events can obviously both occur together so they are not mutually exclusive. They are, however, ***independent***. That is to say, whether an item is defective or acceptable does not affect the probability of it being tested.

There are also other kinds of independent events. If you toss a coin once and then again a second time, the outcome of the second test is independent of the results of the first one. The results of any third or subsequent test are also independent of any previous results. The probability of heads on any test is 0.5 even if all the previous tests have resulted in heads.

To work out the probability of two independent events both happening, you use the ***multiplication law***. This can be stated as:

$$P(A \text{ and } B) = P(A) \times P(B) \text{ if A and B are independent events}$$

Again this result is true for any number of independent events. So:

$$P(A \text{ and } B \text{ ... and } N) = P(A) \times P(B) \text{ ... } \times P(N)$$

Consider the example above. For any item:

Probability that it is defective:	P(D) = 0.2
Probability that it is acceptable:	P(A) = 0.8
Probability that it is checked:	P(C) = 0.1
Probability that it is not checked:	P(N) = 0.9

Using the multiplication law to calculate the probability that an item is both defective and not checked, we get:

$$P(D \text{ and } N) = 0.2 \times 0.9 = 0.18$$

The probabilities of the other combinations of independent events can also be calculated.

$$P(D \text{ and } C) = 0.2 \times 0.1 = 0.02$$

$$P(A \text{ and } N) = 0.8 \times 0.9 = 0.72$$

$$P(A \text{ and } C) = 0.8 \times 0.1 = 0.08$$

Example 1

A machine produces two batches of items. The first batch contains 1,000 items of which 20 are damaged. The second batch contains 10,000 items of which 50 are damaged. If one item is taken from each batch, what is the probability that both items are defective?

For the item from the first batch:

Probability that it is defective: $P(D_1) = \frac{20}{1{,}000} = 0.02$

For the item taken from the second batch:

Probability that it is defective: $P(D_2) = \frac{50}{10{,}000} = 0.005$

Since these two probabilities are independent:

$$P(D_1 \text{ and } D_2) = P(D_1) \times P(D_2) = 0.02 \times 0.005 = 0.0001$$

Example 2

A card is drawn at random from a well shuffled pack of playing cards. What is the probability that the card is a heart? What is the probability that the card is a three? What is the probability that the card is the three of hearts?

Probability of a heart: $P(H) = \frac{13}{52} = \frac{1}{4}$

Probability of a three: $P(3) = \frac{4}{52} = \frac{1}{13}$

Probability of the three of hearts, since the suit and the number of a card are independent:

$$P(H \text{ and } 3) = P(H) \times P(3) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$$

Example 3

A die is thrown three times. What is the probability of one or more sixes in these three throws?

Probability of no six in first throw $= \frac{5}{6}$

Similarly, probability of no six in second or third throw $= \frac{5}{6}$

The result of each throw is independent, so:

Probability of no six in all three throws $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

Since no sixes and one or more sixes are mutually exclusive and cover all possibilities:

Probability of one or more sixes $= 1 - \frac{125}{216} = \frac{91}{216}$

Distinguishing the Laws

Although the above laws of probability are not complicated, you must think carefully and clearly when using them. Remember that:

- events must be ***mutually exclusive*** before you can use the ***addition law***; and
- they must be ***independent*** before you can use the ***multiplication law***.

Another matter about which you must be careful is the listing of equally likely outcomes. Be sure that you list all of them. For example, we can list the possible results of tossing two coins:

First Coin	*Second Coin*
Heads	Heads
Tails	Heads
Heads	Tails
Tails	Tails

There are four equally likely outcomes. Do not make the mistake of saying, for example, that there are only two outcomes (both heads or not both heads) – you must list ***all*** the possible outcomes. (In this case "not both heads" can result in three different ways, so the probability of this result will be higher than "both heads".)

In this example, the probability that there will be one heads and one tails (heads-tails, or tails-heads) is 0.5. This is a case of the addition law at work:

the probability of heads-tails (¼) *plus* the probability of tails-heads (¼).

Putting it another way, the probability of different faces is equal to the probability of the same faces – in both cases 0.5.

D. CONDITIONAL PROBABILITY

The probability of an event A occurring where that event is conditional upon event B having also occurred is expressed as:

$$P(A|B)$$

The events A and B must be causally linked in some way for A to be conditional upon B. For example, we noted earlier the case where a car owner's ability to drive to work in his car (event A) is dependent on him being able to start the car (event B). There is self evidently a causal link between the two events. However, if we consider the case him being able to drive to work given that he is married, there is no necessary link between the two events – and there is no conditional probability.

In a case involving equally likely outcomes for both events (i.e. both events A and B may or may not occur), the conditional probability of event A occurring, given that B also occurs, is calculated by the following formula:

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} \quad \textit{where } n = \text{the number of possible outcomes.}$$

One approach to determining probability in such circumstances is to construct a table of the possible outcomes – sometimes called a ***contingency table***.

Consider the case of 100 candidates attempting their driving test for the first time. 80 of the candidates pass and 50 of them have had 10 or more lessons. Of those who have had 10 or more lessons, 45 of them passed. What is the probability of passing at the first attempt if you have had only 6 lessons?

We can construct a table which identifies the possible outcomes as follows:

	Passed	Failed	Total
10+ lessons	45		50
−10 lessons			
Total	80		100

Make sure you understand the construction of this table:

- We have listed the possible outcomes of event A along the top – these are either to have passed or failed the test. Thus, the columns will show the number of outcomes for either possibility.
- We have listed the possible outcomes of event B along the side – these are either to have had 10 or more lessons, or less than 10 lessons. Thus, the rows will show the number of outcomes for either possibility.
- We have entered in the information that we know – that there are a total of 100 possible outcomes, that the total number of outcomes in which event A was "passed" is 80, that the total number of outcomes in which event B is "had 10 or more lessons" is 50, and that the total number of outcomes in which event A was "passed" and event B was "had 10 or more lessons" is 45.

We can complete the rest of the table by subtraction as follows:

	Passed	Failed	Total
10+ lessons	45	5	50
–10 lessons	35	15	50
Total	80	20	100

We can now work out the total outcomes applicable to the question "what is the probability of passing at the first attempt if you have had only 6 lessons?".

Remember that event A is "passed" and event B is "having had less than 10 lessons". Therefore:

- The total number of outcomes of B = 50
- The total number of outcomes of both A and B = 35

Thus the probability is:

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} = \frac{35}{50} = 0.7$$

We could consider the question "what is the probability of failing at the first attempt if you have had more than 10 lessons?". Here event A is "failed" and event B is "having had more than 10 lessons". The calculation is:

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} = \frac{5}{50} = 0.1$$

See if you can work out the answer to the following questions using the above table:

What is the probability of failing at the first attempt if you have only had 8 lessons?

Remember to be clear about which is event A and event B, then read off the total number of outcomes for event B and for event A and B together, and then apply the formula.

Event A is "failed" and event B is "having less than 10 lessons". The calculation, then is:

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} = \frac{15}{50} = 0.3$$

D. TREE DIAGRAMS

A compound experiment, i.e. one with more than one component part, may be regarded as a sequence of similar experiments. For example, the rolling of two dice can be considered as the rolling of one followed by the rolling of the other; and the tossing of four coins can be thought of as tossing one after the other. A tree diagram enables us to construct an exhaustive list of mutually exclusive outcomes of a compound experiment.

Furthermore, a tree diagram gives us a pictorial representation of probability.

- By ***exhaustive***, we mean that every possible outcome is considered.
- By ***mutually exclusive*** we mean, as before, that if one of the outcomes of the compound experiment occurs then the others cannot.

In the following examples, we shall consider the implications of ***conditional probability*** – the probability of one event occurring if another event has also occurred.

Example 1

The concept can be illustrated using the example of a bag containing five red and three white billiard balls. If two are selected at random without replacement, what is the probability that one of each colour is drawn?

Let: R indicate a red ball being selected

W indicate a white ball being selected

When the first ball is selected, the probabilities are:

$$P(R) = \frac{5}{8}$$

$$P(W) = \frac{3}{8}$$

When the second ball is selected, there are only seven balls left and the mix of red and white balls will be determined by the selection of the first ball – i.e. the probability of a red or a white ball being selected is ***conditional*** upon the previous event. Thus:

If the first ball selected was red, there will be four red and three white balls left. The probabilities for the selection of the second ball are then:

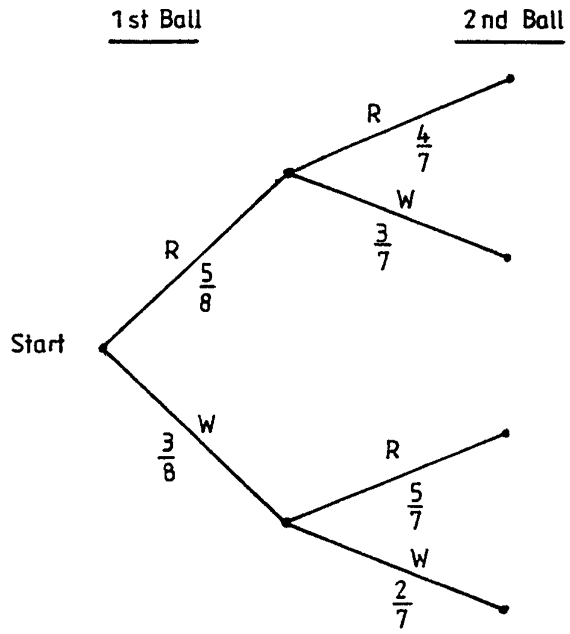
$$P(R) = \frac{4}{7}, \text{ and } P(W) = \frac{3}{7}$$

If the first ball selected was white, there will be five red and two white balls left. The probabilities for the selection of the second ball are then:

$$P(R) = \frac{5}{7}, \text{ and } P(W) = \frac{2}{7}$$

This process of working out the different conditional probabilities can be very complicated and it is easier to represent the situation by means of a tree diagram as in Figure 11.1. The probabilities at each stage are shown alongside the branches of the tree.

Figure 11.1: Example 1 – Tree diagram



The diagram is constructed as follows, working from left to right.

- At the start we take a ball from the bag. This ball is either red or white so we draw two branches labelled R and W, corresponding to the two possibilities. We then also write on the branch the probability of the outcome of this simple experiment being along that branch.
- We then consider drawing a second ball from the bag. Whether we draw a red or a white ball the first time, we can still draw a red or a white ball the second time, so we mark in the two possibilities at the end of each of the two branches of our existing tree diagram. We enter on these second branches the conditional probabilities associated with them. Thus, on the uppermost branch in the diagram we must insert the probability of obtaining a second red ball given that the first was red.

Each complete branch from start to tip represents one possible outcome of the compound experiment and each of the branches is mutually exclusive. We can see that there are four different mutually exclusive outcomes possible, namely RR, RW, WR and WW.

To obtain the probability of a particular outcome of the compound experiment occurring, we multiply the probabilities along the different sections of the branch, using the general multiplication law for probabilities. We thus obtain the probabilities shown in Table 11.1. The sum of the probabilities should add up to 1, as we know one or other of these mutually exclusive outcomes is certain to happen.

Table 11.1: Example 1 – Probabilities of outcomes from compound (conditional) events

Outcome	Probability
RR	$\frac{5}{8}\times\frac{4}{7}=\frac{20}{56}$
RW	$\frac{5}{8}\times\frac{3}{7}=\frac{15}{56}$
WR	$\frac{3}{8}\times\frac{5}{7}=\frac{15}{56}$
WW	$\frac{3}{8}\times\frac{2}{7}=\frac{6}{56}$
Total	1

Example 2

A bag contains three red balls, two white balls and one blue ball. Two balls are drawn at random (without replacement). Find the probability that:

(a) Both white balls are drawn.

(b) The blue ball is not drawn.

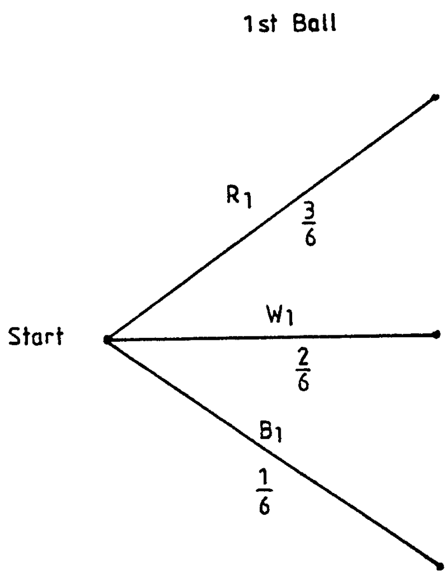
(c) A red then a white are drawn.

(d) A red and a white are drawn.

To solve this problem, let us build up a tree diagram.

Stage 1 will be to consider the situation when the first ball is drawn:

Figure 11.2: Example 2 – Tree diagram of first event



We have given the first ball drawn a subscript of 1 – for example, red first = R_1. We shall give the second ball drawn a subscript of 2.

The probabilities in the first stage will, then, be:

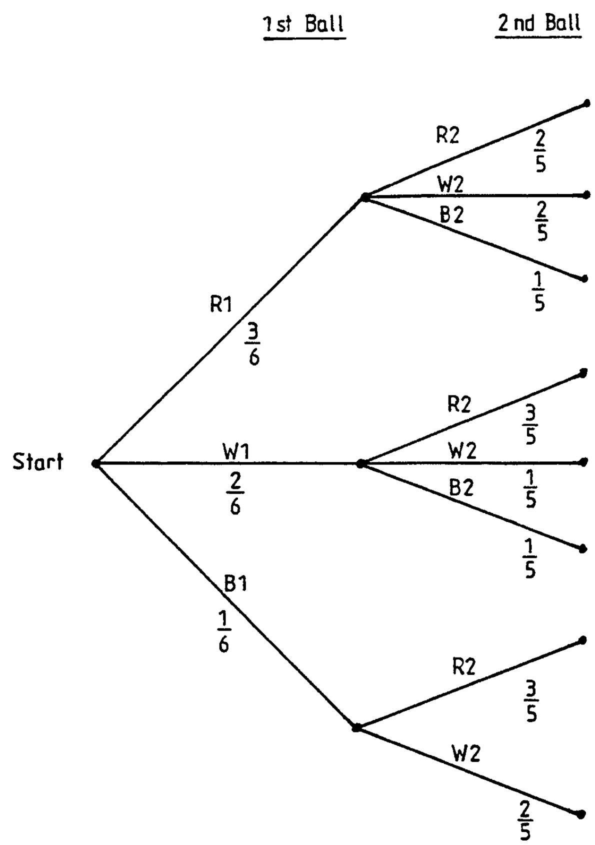
$$P(R_1) = \frac{\text{No. of red balls}}{\text{Total no. of balls}} = \frac{3}{6}$$

$$P(W_1) = \frac{2}{6}$$

$$(P(B_1) = \frac{1}{6}$$

The possibilities in stage 2 may now be mapped on the tree diagram. When the second ball is drawn, the probability of it being red, white or blue is conditional on the outcome of the first event. Thus, note there was only one blue ball in the bag, so if we picked a blue ball first then we can have only a red or a white second ball. Also, whatever colour is chosen first, there are only five balls left as we do not have replacement.

Figure 11.3: Example 2 – Tree diagram of first and second events



We can now list all the possible outcomes, with their associated probabilities.

Table 11.2: Example 2 – Probabilities of outcomes from compound (conditional) events

Outcome	Probability
R_1R_2	$\frac{3}{6}\times\frac{2}{5}=\frac{1}{5}$
R_1W_2	$\frac{3}{6}\times\frac{2}{5}=\frac{1}{5}$
R_1B_2	$\frac{3}{6}\times\frac{1}{5}=\frac{1}{10}$
W_1R_2	$\frac{2}{6}\times\frac{3}{5}=\frac{1}{5}$
W_1W_2	$\frac{2}{6}\times\frac{1}{5}=\frac{1}{15}$
W_1B_2	$\frac{2}{6}\times\frac{1}{5}=\frac{1}{15}$
B_1R_2	$\frac{1}{6}\times\frac{3}{5}=\frac{1}{10}$
B_1W_2	$\frac{1}{6}\times\frac{2}{5}=\frac{1}{15}$
Total	1

It is possible to read off the probabilities we require from Table 11.2.

(a) Probability that both white balls are drawn:

$$P(W_1W_2) = \frac{1}{15}$$

(b) Probability the blue ball is not drawn:

$$P(R_1R_2) + P(R_1W_2) + P(W_1R_2) + P(W_1W_2) = \frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{15}=\frac{2}{3}$$

(c) Probability that a red then a white are drawn:

$$P(R_1W_2) = \frac{1}{5}$$

(d) Probability that a red and a white are drawn:

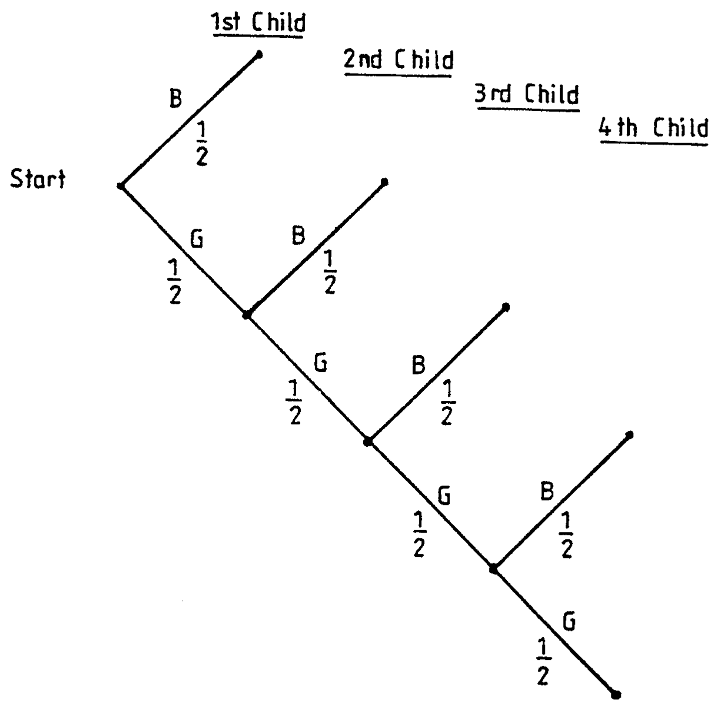
$$P(R_1W_2) + P(W_1R_2) = \frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

Example 3

A couple go on having children, to a maximum of four, until they have a son. Draw a tree diagram to find the possible families' size and calculate the probability that they have a son.

We can assume that any one child is equally likely to be a boy or a girl, i.e. P(B) = P(G) = 0.5. Note that once they have produced a son, they do not have any more children. The tree diagram will be as in Figure 11.4.

Figure 11.4: Example 3 – Tree diagram



The table of probabilities is as follows.

Table 11.3: Example 3 – Probabilities of outcomes

Possible Families	Probability
1 Boy	$\frac{1}{2}$
1 Girl, 1 Boy	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
2 Girls, 1 Boy	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
3 Girls, 1 Boy	$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$
4 Girls	$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$
Total	1

The probability that the family will have a son before their fourth child is therefore:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

E. SAMPLE SPACE

You need a clear head to perform probability calculations successfully. It helps, then, to have some diagrammatic form of representation. We have seen the value of tree diagrams in the last section, and now we consider another method of drawing the possible outcomes of an event.

When we, say, toss a coin three times and note the outcome, we are performing a statistical experiment. If we make a list of all the possible outcomes of our experiment, we call this a ***sample space***. The sample space in the coin-tossing experiment is:

H H H	H T T
T H H	T H T
H T H	T T H
H H T	T T T

where, for example, T H H means that on the first toss we obtained a tail, on the second a head, and on the third a head.

Let us consider another example.

Suppose we have 5 people – A, B, C, D, E – and we wish to select a random sample of 2 for interview – i.e. each of A, B, C, D, E must have the same chance of being chosen. What is the sample space, i.e. the list of all possible different samples?

The sample space is:

A B	B C	C D	D E
A C	B D	C E	
A D	B E		
A E			

In this example the order of the sample, i.e. whether we choose A followed by B or B followed by A, is not relevant as we would still interview both A and B.

Having written down our sample space, we might be interested in a particular section of it.

For instance, in our first example we might want to see, in how many of the outcomes, we obtained only one head – i.e. we are interested in the event of obtaining exactly one head. Looking back at the sample space, we see that there are three outcomes making up this event. If we have a fair coin, the probability of obtaining a head or a tail at any toss is ½ and all the outcomes in the sample space are equally likely. There are eight outcomes in all, so we can now deduce that:

Probability of obtaining exactly one head in 3 tosses of a fair coin

$$= \frac{\text{No. of outcomes in sample space with one head}}{\text{Total no. of outcomes in sample space}}$$

$$= \frac{3}{8}$$

Alternatively, we could write:

$$\text{P(Event A)} = \frac{\text{No. of outcomes in A}}{\text{Total no. of outcomes}}$$

$$= \frac{3}{8}$$

Now consider the example of an experiment involving the rolling of one fair die. What is the probability of obtaining:

(a) a score greater than 3

(b) an odd number

(c) both a score greater than 3 and an odd number

The sample space is a listing of all the possible outcomes as follows:

1, 2, 3, 4, 5, 6

The total number of possible outcomes is 6, and it is now easy to read off the probabilities from the sample space:

(a) a score greater than 3 – three possibilities: $P = \frac{3}{6} = \frac{1}{2}$

(b) an odd number – three possibilities: $P = \frac{3}{6} = \frac{1}{2}$

(c) both a score greater than 3 and an odd number – one possibility: $P = \frac{1}{6}$

F. EXPECTED VALUE

Probabilities may also be used to assist the decision-making process in business situations. Before a new undertaking is started, for example, it may be possible to list or categorise all the different possible outcomes. On the basis of past experience it may also be possible to attribute probabilities to each outcome. These figures can then be used to calculate an expected value for the income from the new undertaking.

Example

Suppose a company is considering opening a new retail outlet in a town where it currently has no branches. On the basis of information from its other branches in similar towns, it can categorise the possible revenue from the new outlet into categories of high, medium or low. Table 11.4 shows the probabilities which have been estimated for each category. You will note that the probabilities sum to 1.

Table 11.4: Probabilities of likely revenue returns

Likely Revenue	Probability
High	0.4
Medium	0.5
Low	0.1

Using these figures it is possible to calculate the "expected value" for the revenue from the new branch.

The expected value is an average for the revenue, weighted by the probabilities. It is the sum of each value for the outcome multiplied by its probability, calculated by the following formula:

$$\text{Expected value} = \sum Px$$

where x is the value of the outcome.

In our example, if we give values to the outcomes of each category – i.e. figures for the high, medium and low revenues – and then multiply each by its probability and sum them, we get an average revenue return for new branches. This expected revenue can then be compared with the outlay on the new branch in order to assess the level of profitability.

Table 11.5: Expected value of revenue returns

Category	Revenue *(£000)*	Probability	Prob. × Rev. *(£000)*
High	5,000	0.4	2,000
Medium	3,000	0.5	1,500
Low	1,000	0.1	100
		Total:	£3,600

The concept of expected value is used to incorporate uncertainty into the decision-making process.

Practice Questions 1

1. If three coins are thrown, what is the probability that all three will show tails?

2. (a) (i) What is the probability of getting a total of six points in a simultaneous throw of two dice?

   (ii) What is the probability of getting at least one six from a simultaneous throw of two dice?

   (b) Out of a production batch of 10,000 items, 600 are found to be of the wrong dimensions. Of the 600 which do not meet the specifications for measurement, 200 are too small. What is the probability that, if one item is taken at random from the 10,000:

   (i) It will be too small?

   (ii) It will be too large?

   If two items are chosen at random from the 10,000, what is the probability that:

   (iii) Both are too large?

   (iv) One is too small and one too large?

3. (a) Calculate, when three dice are thrown together, the probability of obtaining:

   (i) three sixes;

   (ii) one six;

   (iii) at least one six.

   (b) Calculate the probability of drawing in four successive draws the ace and king of the hearts and diamonds suits from a pack of 52 playing cards.

(c) Calculate the probability that three marbles drawn at random from a bag containing 24 marbles, six of each of four colours (red, blue, green and white), will be:

(i) all red;

(ii) all of any one of the four colours.

4. An analysis of 72 firms which had introduced flexi-time working showed that 54 had used some form of clocking-in system whilst the rest relied on workers completing their own time sheets. 60 of the firms found that output per worker rose and in all other firms it fell. Output fell in only 3 of the firms using worker completed forms.

Construct a contingency table and then:

(a) Calculate the probability of output per worker falling if an automatic clocking-in system is introduced.

(b) Determine under which system it is more probable that output per worker will rise.

5. The probability that machine A will be performing a useful function in five years' time is 0.25, while the probability that machine B will still be operating usefully at the end of the same period is 0.33.

Find the probability that, in five years' time:

(a) Both machines will be performing a useful function.

(b) Neither will be operating.

(c) Only machine B will be operating.

(d) At least one of the machines will be operating.

6. Three machines, A, B and C, produce respectively 60%, 30% and 10% of the total production of a factory. The percentages of defective production for each machine are respectively 2%, 4% and 6%.

(a) If an item is selected at random, find the probability that the item is defective.

(b) If an item is selected at random and found to be defective, what is the probability that the item was produced by machine A?

7. Using a sample space diagram, calculate the probability of getting, with two throws of a die:

(a) A total score of less than 9.

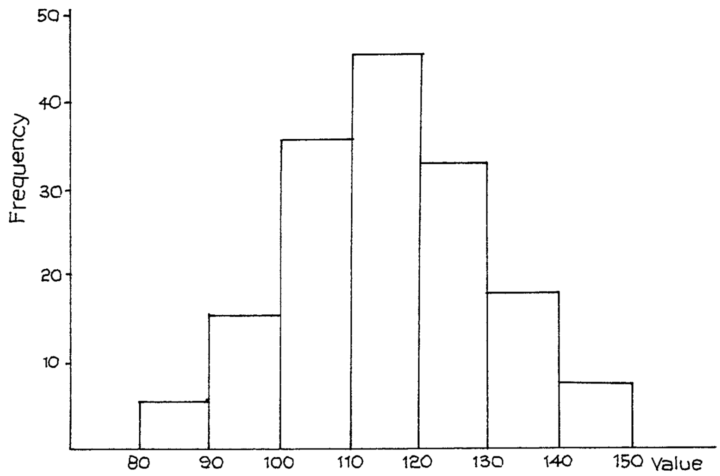
(b) A total score of 9.

Now check your answers with the ones given at the end of the unit.

G. THE NORMAL DISTRIBUTION

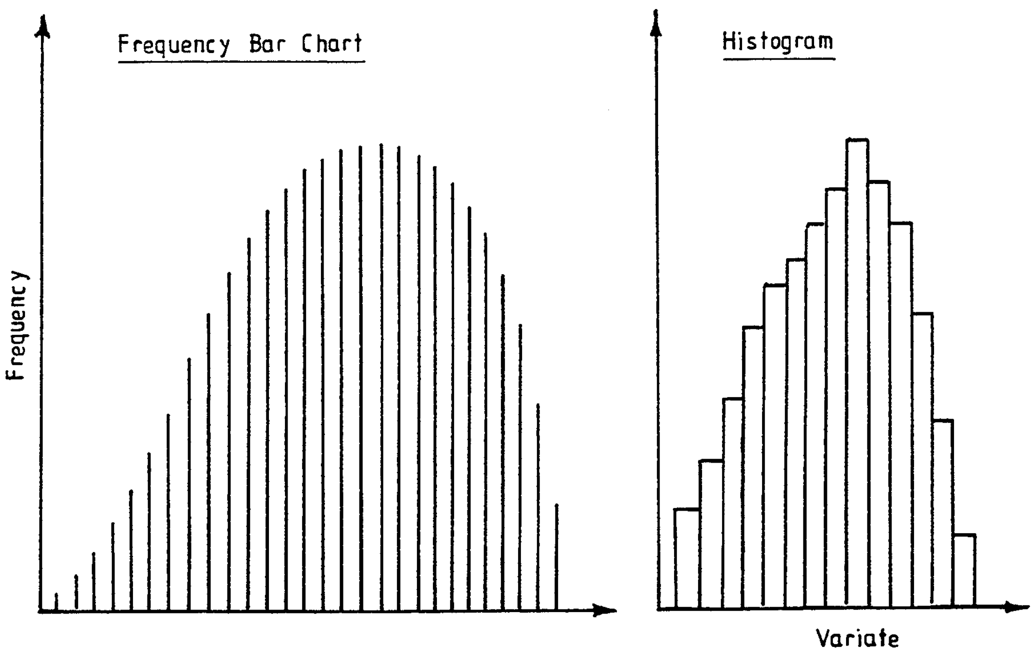
Earlier in the course, we considered various graphical ways of representing a frequency distribution. We considered a frequency dot diagram, a bar chart, a polygon and a frequency histogram. For example, a typical histogram will take the form shown in Figure 11.5. You can immediately see from this diagram that the values in the centre are much more likely to occur than those at either extreme.

Figure 11.5: Typical frequency histogram



Consider now a continuous variable in which you have been able to make a very large number of observations. You could compile a frequency distribution and then draw a frequency bar chart with a very large number of bars, or a histogram with a very large number of narrow groups. Your diagrams might look something like those in Figure 11.6.

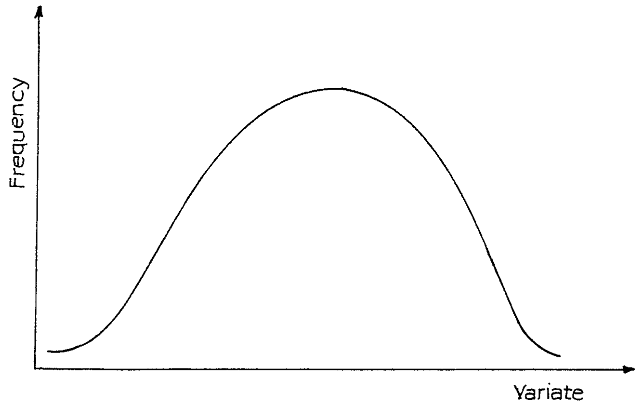
Figure 11.6: Frequency bar chart and histogram



If you now imagine that these diagrams relate to a ***relative frequency distribution*** and that a smooth curve is drawn through the tops of the bars or rectangles, you will arrive at the idea of a ***frequency curve***.

We can then simplify the concept of a frequency distribution by illustrating it without any actual figures as a frequency curve which is just a graph of frequency against the variate:

Figure 11.7: Frequency curve

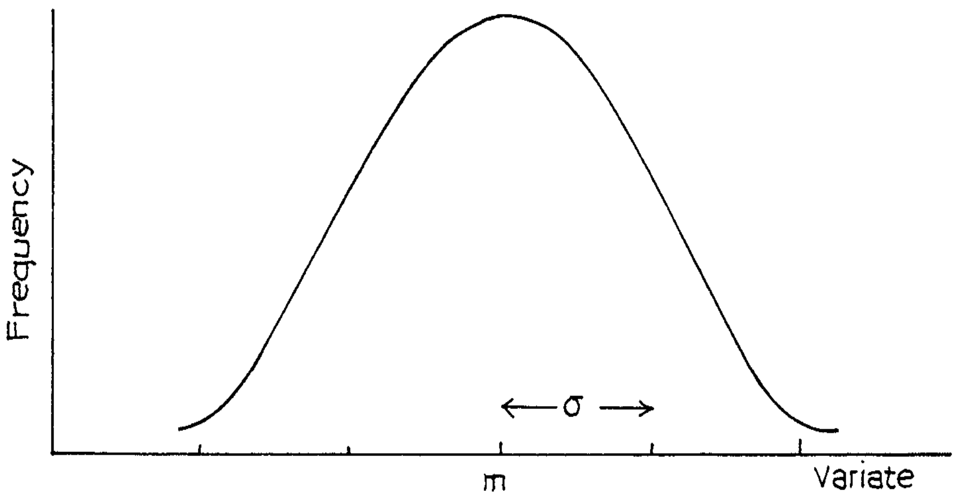


The "***normal***" or "***Gaussian***" distribution was discovered in the early 18th century and is probably the most important distribution in the whole of statistical theory. This distribution seems to represent accurately the random variation shown by natural phenomena – for example:

- heights of adult men from one race
- weights of a species of animals
- the distribution of IQ levels in children of a certain age
- weights of items packaged by a particular packing machine
- life expectancy of light bulbs

The typical shape of the normal distribution is shown in Figure 11.8. You will see that it has a central peak (i.e. it is unimodal) and that it is symmetrical about this centre. The mean of this distribution is shown as "m" on the diagram, and is located at the centre. The standard deviation, which is usually denoted by "σ", is also shown.

Figure 11.8: The normal distribution



Properties of the Normal Distribution

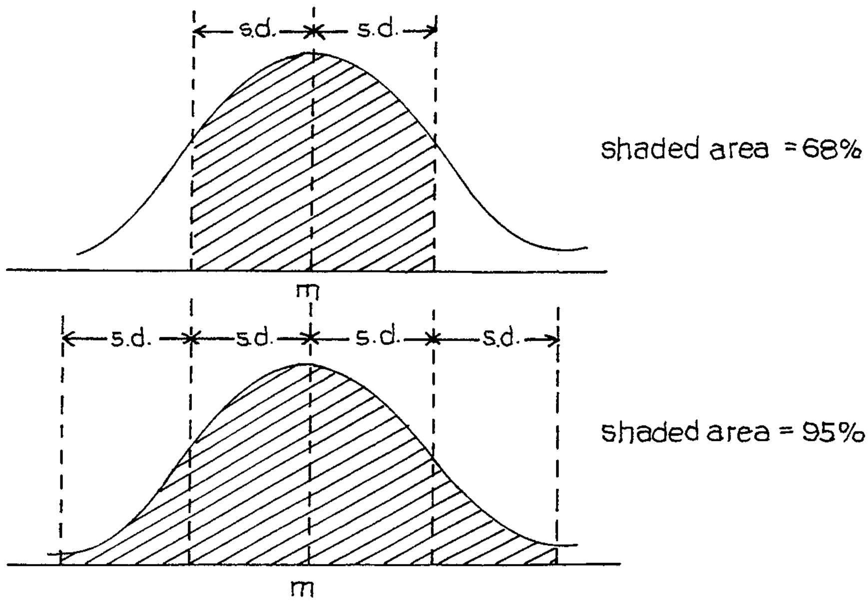
The normal distribution has a number of interesting properties. These allow us to carry out certain calculations on the distribution.

For normal distributions, we find that:

- exactly 68% of the observations are within ±1 standard deviation of the mean; and
- exactly 95% are within ±2 standard deviations of the mean.

We can illustrate these properties on the curve itself:

Figure 11.9: Properties of the normal distribution



These figures can be expressed as probabilities. For example, if an observation x comes from a normal distribution with mean m and standard deviation σ, the probability that x is between (m – σ) and (m + σ) is:

$$P(m - \sigma < x < m + \sigma) = 0.68$$

Also

$$P(m - 2\sigma < x < m + 2\sigma) = 0.95$$

Using Tables of the Normal Distribution

Because the normal distribution is perfectly symmetrical, it is possible to calculate the probability of an observation being within *any* range, not just (m – σ) to (m + σ) and (m – 2σ) to (m + 2σ). These calculations are complex, but fortunately, there is a short cut available.

There are tables which set out these probabilities.

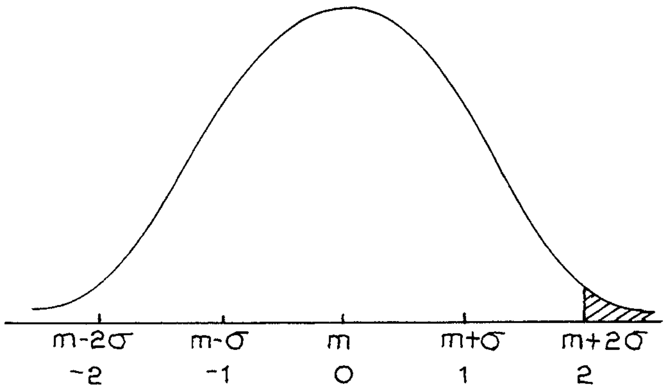
We provide one such table on the next page. This relates to the area in the tail of the normal distribution.

AREAS IN TAIL OF THE NORMAL DISTRIBUTION

$\frac{(x - \mu)}{\sigma}$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
3.0	.00135									

Note that the mean, here, is denoted by the symbol "μ" (the Greek letter for m, pronounced "mu"). Thus, the first column in the table represent the position of an observation, x, in relation to the mean (x – μ), expressed in terms of the standard deviation by dividing it by σ.

Figure 11.10: Proportion of the distribution in an area under the curve of a normal distribution



The area under a section of the curve represents the proportion of observations of that size in relation to the total number of observations. For example, the shaded area shown in Figure 11.10 represents the chance of an observation being greater than $m + 2\sigma$. The vertical line which defines this area is at $m + 2\sigma$.

Looking up the value 2 in the table gives the proportion of the normal distribution which lies outside of that value:

$$P(x > m + 2\sigma) = 0.02275$$

which is just over 2%.

Similarly, $P(x > m + 1\sigma)$ is found by looking up the value 1 in the table. This gives:

$$P(x > m + 1\sigma) = 0.1587$$

which is nearly 16%.

You can extract any value from $P(x = m)$ to $P(x > m + 3\sigma)$ from this table. This means that you can find the area in the tail of the normal distribution wherever the vertical line is drawn on the diagram.

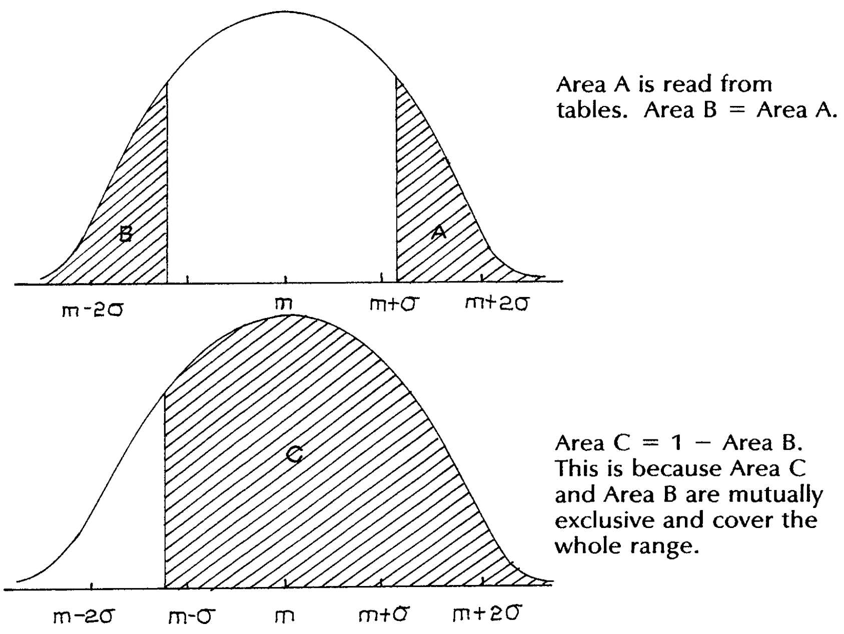
Negative distances from the mean are not shown in the tables. However, since the distribution is symmetrical, it is easy to calculate these.

$$P(x < m - 5\sigma) = P(x > m + 5\sigma)$$

So $\quad P(x > m - 5\sigma) = 1 - P(x < m - 5\sigma)$

This is illustrated in Figure 11.11.

Figure 11.11: Using the symmetry of the normal distribution to determine the area under the curve

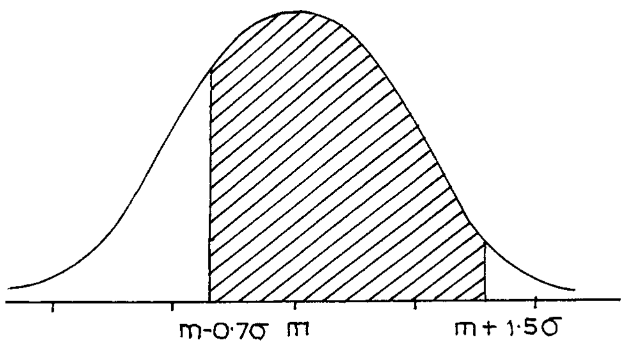


Further Probability Calculations

It is possible to calculate the probability of an observation being in the shaded area shown in Figure 11.12, using values from the tables. This represents the probability that x is between m – 0.7σ and m + 1.5σ: i.e.

$P(m - 0.7\sigma < x < m + 1.5\sigma)$

Figure 11.12: Probability of an observation being in a particular area of the normal distribution



First find $P(x > m + 1.5\sigma) = 0.0668$ from the table.

Then find $P(x < m - 0.7\sigma)$ or $P(x > m + 0.7\sigma)$

$= 0.2420$ since the distribution is symmetrical.

The proportion of the area under the curve which is shaded in the Figure, then, is:

$1 - 0.0668 - 0.2420 = 0.6912$

Hence $P(m - 0.7\sigma < x < m + 1.5\sigma) = 0.6912$.

Example

A production line produces items with a mean weight of 70 grams and a standard deviation of 2 grams. Assuming that the items come from a normal distribution, find the probability that an item will weigh 65 grams or less.

65 grams is 5 grams below the mean. Since the standard deviation is 2 grams, this is 2.5 standard deviations below the mean.

Let x be the weight of an individual item. Therefore:

$$P(x < 65) = P(x < m - 2.5\sigma)$$

$$= P(x > m = 2.5\sigma)$$

$$= 0.00621 \text{ from the tables.}$$

Now find the probability that an item will weigh between 69 and 72 grams.

69 grams is 0.5 standard deviations below the mean, and 72 grams is 1 standard deviation above the mean. Therefore we need to find $P(m - 0.5\sigma < x < m + \sigma)$:

$$P(x > m + \sigma) = 0.1587$$

$$P(x < m - 0.5\sigma) = P(x > m + 0.5\sigma) = 0.3085$$

So, $P(m - 0.5\sigma < x < m + \sigma) = 1 - 0.1587 - 0.3085$

or $P(69 < x < 72) = 0.5328$

Practice Questions 2

1. A factory which makes batteries tests some of them by running them continuously. They are found to have a mean life of 320 hours and a standard deviation of 20 hours. The shape of this distribution is similar to the normal distribution. By assuming that this distribution is normal, estimate:

   (a) The probability that the battery will last for more than 380 hours.

   (b) The probability that it will last for less than 290 hours.

   (c) The probability that it will last for between 310 and 330 hours.

   (d) The value below which only 1 in 500 batteries will fall.

2. A machine produces 1,800 items a day on average. The shape of the distribution of items produced in one day is very similar to a normal distribution with a standard deviation of 80 items. On any one day:

   (a) If more than 2,000 items are produced, the workers get a bonus. What is the probability that a bonus will be earned?

   (b) When less than 1,700 items are produced, overtime is available. What is the probability that overtime will be worked?

   (c) What is the probability that the workforce neither gets a bonus nor works overtime?

Now check your answers with the ones given at the end of the unit.

ANSWERS TO PRACTICE QUESTIONS

Practice Questions 1

1. The probability of tails is 0.5 for each coin. The events are independent. Therefore:

$$P(3T) = \tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}=\tfrac{1}{8}$$

2. (a) (i) Six points can be obtained from the following combinations:

1st die	*2nd die*
1	5
2	4
3	3
4	2
5	1

Since each die can fall with any of its faces uppermost, there are a total of 36 possible combinations, i.e. $6 \times 6 = 36$. Therefore:

$$P(\text{six points}) = \frac{5}{36}$$

(ii) $P(\text{no six}) = \dfrac{5}{6}\times\dfrac{5}{6}=\dfrac{25}{36}$

$$P(\text{at least one six}) = 1 - P(\text{no six})$$

$$= \frac{11}{36}$$

(b) (i) $P(\text{too small}) = \dfrac{200}{10{,}000}=0.02$

(ii) $P(\text{too large}) = \dfrac{400}{10{,}000}=0.04$

(iii) Note that, if the first item is too large, it will be removed from the calculation of the second item. Therefore:

$$P(\text{both too large}) = P(\text{1st too large}) \times P(\text{2nd too large})$$

$$= 0.04 \times \frac{399}{9{,}999}$$

$$= 0.001596$$

(iv) The first item can be either too large or too small. Therefore:

$$P(\text{one too large, one too small}) = 0.04\times\frac{199}{9{,}999}+0.02\times\frac{399}{9{,}999}$$

$$= 0.001594$$

3. (a) (i) Each die must show a six. Therefore:

$$\text{P(three sixes)} = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

(ii) One die must show a six and the others must not show a six. Any one of the three dice could be the one that shows a six.

No. of ways the die showing six can be chosen = 3

Probability of six on one die = $\left(\frac{1}{6}\right)$

Probability of not obtaining six on the other two dice = $\left(\frac{5}{6}\right)^2$

Therefore:

$$\text{P(one six)} = 3 \times \left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right)^2 = \frac{25}{72}$$

(iii) Probability of obtaining at least one six

= 1 – Probability of not obtaining any sixes

= 1 – Probability that all three dice show 1, 2, 3, 4 or 5

$$= 1 - \left(\frac{5}{6}\right)^3 = 1 - \frac{125}{216} = \frac{91}{216}$$

(b) Required probability $= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49}$

= 0.000003694 to 4 sig. figs

(c) (i) Probability that first is red = $\frac{6}{24}$

Probability that second is red = $\frac{5}{23}$

Probability that third is red = $\frac{4}{22}$

Probability that all three are red = $\frac{6}{24} \times \frac{5}{23} \times \frac{4}{22} = 0.009881$ to 4 sig. figs

(ii) Probability that the three will be all of any one of the four colours

= Probability that all three are red

+ Probability that all three are blue

+ Probability that all three are green

+ Probability that all three are white

$$= 4 \times \frac{6}{24} \times \frac{5}{23} \times \frac{4}{22}$$

= 0.03953 to 4 sig. figs.

4. The contingency table will be as follows:

	Output Up	Output down	Total
Clocking-in system	45	9	54
Worker completion system	15	3	18
Total	60	12	72

(a) The probability of output per worker falling (event A) given that an automatic clocking-in system is used (event B) is:

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} = \frac{9}{54} = \frac{1}{6}$$

(b) The probability of output per worker rising (event A) given that an automatic clocking-in system is used (event B) is:

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} = \frac{45}{54} = \frac{5}{6}$$

The probability of output per worker rising (event A) given that a worker completion system is used (event B) is:

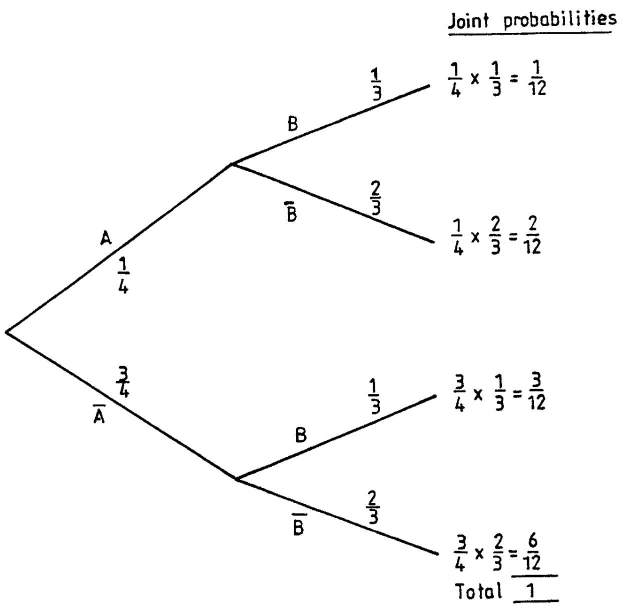
$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} = \frac{15}{18} = \frac{1}{6}$$

The probabilities are the same.

5. *Let:* A = Machine A performing usefully, and $\overline{A}$ = Machine A not performing usefully

B = Machine B performing usefully, and $\overline{B}$ = Machine B not performing usefully.

We can draw a tree diagram to map the situation as follows:



(a) Probability of both machines operating is $\frac{1}{12}$

(b) Probability of neither operating is $\frac{6}{12}$ or $\frac{1}{2}$

(c) Probability of only B operating is $\frac{3}{12}$ or $\frac{1}{4}$

(d) Probability of at least one machine operating is $\frac{1}{12}+\frac{2}{12}+\frac{3}{12}=\frac{1}{2}$

6. *Let:* A = produced on machine A

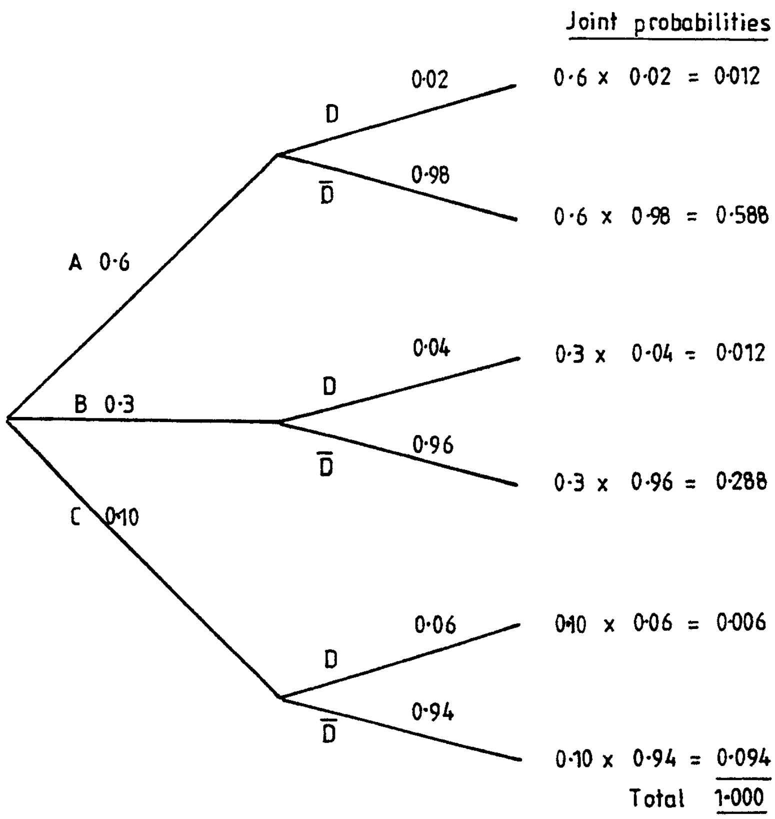
B = produced on machine B

C = produced on machine C

D = defective

$\overline{D}$ = not defective.

The tree diagram for the situation is shown below.



(a) P(D) = 0.012 + 0.012 + 0.006 = 0.03

(b) P(defective that came from machine A) = $\dfrac{\text{Probability of a defective from A}}{\text{Probability of defective}}$

$$= \frac{0.012}{0.03}$$

$$= 0.4$$

7. The sample space diagram sets out all the possible total scores obtainable when two dice are thrown. There are 36 equally likely outcomes:

2nd die \ 1st die:	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(a) 26 outcomes give scores less than 9. Therefore:

$$\text{P(total score less than 9)} = \frac{26}{36} = \frac{13}{18} = 0.722$$

(b) 4 outcomes give a score of 9. Therefore:

$$\text{P(total score of 9)} = \frac{4}{36} = \frac{1}{9} = 0.111$$

Practice Questions 2

1. (a) 380 hours is 60 hours or 3 standard deviations above the mean. Therefore:

P(observation > m + 3σ) = 0.00135 from the tables.

(b) 290 hours is 30 hours or 1.5 standard deviations below the mean. Therefore:

P(observation < m – 1.5σ) = P(observation > m + 1.5σ)

= 0.0668

(c) 310 hours is 0.5 standard deviations below the mean and 330 hours is 0.5 standard deviations above the mean.

P(observation > m + 0.5σ) = 0.3085 from the tables

Similarly, P(observation < m – 0.5σ) = 0.3085

Therefore:

P(m – 0.5σ < observation < m + 0.5σ) = P(310 < observation < 330)

= 1 – 0.3085 – 0.3085

= 0.383

Alternatively, using the symmetry of the normal distribution:

$$P(m < \text{observation} < m + 0.5\sigma) = 0.5\sigma - 0.3085 = 0.1915.$$

Hence, $P(m - 0.5\sigma < \text{observation} < m + 0.5\sigma) = 2 \times 0.1915 = 0.383$

(d) Expressing 1 in 500 as a proportion:

$$1 \text{ in } 500 = \frac{1}{500} = 0.002$$

Using the tables in reverse, and looking up the value 0.002 in the body of the tables, the nearest we can get is 0.00199. This is when:

$$\left(\frac{x - \mu}{\sigma}\right) = 2.88$$

Therefore, 1 in 500 batteries will be more than 2.88 standard deviations below the mean.

Answer: $= 320 - 2.88 \times 20$

$= 320 - 57.6 = 262.4$

Therefore 1 in 500 batteries will have a life of less than 262.4 hours.

2. Mean, m = 1,800

Standard deviation, σ = 80

(a) In a normal distribution, 2,000 items is:

$$\frac{(2{,}000 - 1{,}800)}{80} = 2.5 \text{ standard deviations from the mean}$$

Therefore, the probability of 2,000 items being produced is:

$$P(x > m + 2.5\sigma) = 0.00621$$

(b) In a normal distribution, 1,700 items is:

$$\frac{(1{,}700 - 1{,}800)}{80} = -1.5 \text{ standard deviations from the mean}$$

Therefore, the probability of 1,700 items being produced is:

$$P(x < m - 1.25\sigma) = P(x > m + 1.25\sigma)$$

$$= 0.1056$$

(c) Probability of no bonus and no overtime

$= 1 - 0.1056 - 0.00621$

$= 0.88819$